The Mathematics Teacher

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Contemporary applications of mathematics[1]

RICHARD S. BURINGTON, *Chief Mathematician, Bureau of Ordnance, Navy Department.* *Mathematics plays a fundamental role in helping to solve technological problems. This article supplies a glimpse into how mathematics helps solve these problems. It warns teachers that ". . . the coming generations must be better equipped to handle . . . technological problems. . . ."*

Introduction

How does mathematics appear in the real world today? What sorts of mathematics play dominant roles in contemporary industrial applications? How do the applications of mathematics arise? To what extent can it be said that mathematics proves things about the real world? What is the meaning of these considerations in the educational world? How should these considerations affect the educational policies of our secondary schools and our colleges and universities? It is to such questions as these that I direct my attention.

Doubtlessly you have been hearing much about modern algebra and statistics and other equally important realms of mathematics. You have been experiencing a considerable stimulus through these excursions away from the routine of your work. And the impact of this stimulus on your teaching of mathematics will no doubt be impressive.

I shall begin by painting for you a rather general sketch of the sort of mathematics applications I observe as I go about my work, which at present lies in the general field of research and development leading to new equipment and systems. Perhaps my discussion will prove useful to you as teachers of mathematics. Specifically, I hope that those of you who plan curricula and advise students in the secondary schools will go away with added ammunition for your campaigns to give our young people the best possible preparation for a life of real service.

How mathematical applications arise in the real world

The applications of mathematics in the world at large arise principally because someone (or group) is trying to design or build a structure or machine; or because someone wants to understand, describe, explain, or predict the behavior of a phenomena of nature, a machine, or a specific situation in the world of affairs and men. The mathematics, as may be used or needed, arise then from problems in the physical, economic, sociological, or biological fields.

In the world of applications mathematics *per se* seldom arises primarily as a subject to be studied in its own right except as such mathematics may appear as abstractions from the fields of application requiring some sort of solution. The actual research and development of mathematics as such is done largely by professional re-

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The views expressed in this paper are those of the author and are not necessarily those of the Navy Department.

search mathematicians. Research mathematicians oftentimes gain their stimuli and hunches from some field of physical or economic science, and they are frequently stimulated by new discoveries and developments. Thus, the realization of new techniques in electronics in part has led to the rapid development of computing engines of great capacity, computing centers, and an array of real research problems in numerical analysis.

In mathematics applications one is usually dealing with a physical or economic situation. In so doing one uses the language, the techniques, the logics of mathematics. Such applications commonly involve one or more of the following topics:

1. calculations;
2. manipulation of mathematical relationships;
3. graphical analysis;
4. adaptations of existing mathematical theories to the particular subject under consideration, resulting in solutions desired to certain problems;
5. probability and statistical considerations;
6. construction of a theory or model of a physical or economic field of interest, making use of major principles underlying the structural development of a mathematics;
7. development of a mathematical discipline or methodology to supply the needs of the physical or economic field under study. In other words, new mathematical techniques may have to be invented to solve the problems encountered in the application.

Generally speaking, the applications of mathematics encountered in industry and in research and development work seldom present themselves solely as some one branch of knowledge. Rather, such applications appear as problems in overlapping fields that perhaps involve pertinent branches of physical science, engineering, business, human relations, and the like, as well as mathematics. In all such applications a model of the situation under study is made. Mathematical reasoning is used to aid in the construction of the model, and the solution of the desired problem may require the use of mathematical techniques.

The types of mathematics encountered might be classified in part as involving, to varying degrees, the subject matter of arithmetic, algebra, geometry, trigonometry, calculus, differential equations, harmonic analysis, . . . , mathematical statistics, . . . , as organized and taught in our colleges and universities. Likewise, the subject matter encountered involves that of physics, chemistry, mechanics, the properties of materials, and so on, as they pertain to the underlying field of applications. But the actual applications in the real world usually overlap several such specialized branches of mathematics and technology.

The origin of physical theories

Some consideration of the manner in which a theory of a typical physical phenomenon may evolve will throw light on the manner in which mathematics is used.

As a means for following my discussion you may think of the problem of hurling a large stone from a high cliff into a body of water. We begin by considering the underwater path of the stone. At first glance one might suppose that the familiar equations of motion of a point particle of mass m moving through a given medium would suffice to describe the underwater path of the stone, if the water-entry velocity and entry angle were specified. The mathematical model of the simplified problem would then involve the mass of the stone, the time, t, the gravitational pull of the earth on the stone, and the coordinates x, y, z of the assumed point mass, taken as at the center of the stone. If Newton's laws of motion are assumed, the model might be expressed by the system of differential equations

$$(1) \quad m\ddot{x}=0, \quad m\ddot{y}=0, \quad m\ddot{z}=-mg.$$

The solution of these equations for a specified entry angle and velocity, namely,

$$(2) \qquad x=x(t), \quad y=y(t), \quad z=z(t),$$

must then be compared with actual experiments to determine how well the solution (2) fits the actual underwater path of the stone.

A few well-designed experiments would soon show that, for the specified entry conditions, the actual path is of the form

$$(3) \quad x=X(t)+\epsilon_1, \; y=Y(t)+\epsilon_2, \; z=Z(t)+\epsilon_3,$$

where $X(t)$, $Y(t)$, $Z(t)$ are symbols for the mean of the paths observed and ϵ_1, ϵ_2, ϵ_3 are quantities which measure the spread of the various paths actually observed in the tests.

It would require only a few well chosen experiments with stones to show a considerable discrepancy between the predictions (2) and the experimental data (3). The conclusion would be that the model (1) is inadequate.

Further study would soon lead one to consider the effects of drag on the path of the stone. This might lead to a model having the equations

$$(1') \quad m\ddot{x}=D_1, \quad m\ddot{y}=D_2, \quad m\ddot{z}=-mg+D_3,$$

with corresponding solutions

$$(2') \quad x=x'(t), \quad y=y'(t), \quad z=z'(t).$$

Solutions (2′) must be compared with those of experiment (3). Again, if care is taken, the fit of (2′) and (3) will be disappointing. The model (1′) would still be unsatisfactory.

Further investigation would reveal lifting and turning forces, $F \cdots$, which would need to be included in the model (1′), thus further complicating the theory. The model equations would then be of the form

$$(1'') \quad \begin{aligned} \ddot{x}&=f_1(D, F, \cdots) \\ \ddot{y}&=f_2(D, F, \cdots), \\ \ddot{z}&=f_3(D, F, g, \cdots), \end{aligned}$$

with solutions

$$(2'') \quad x=x''(t), \quad y=y''(t), \quad z=z''(t).$$

A comparison of (2″) with experiment (3) would reveal a closer but not a perfect fit. One would soon be disappointed with the theory and renew the investigation.

Further study would reveal the formation of a cavity surrounding the stone upon water entry. More experiments would need to be performed to disclose the nature of the forces acting on the stone upon entry, formation of the cavity, the behavior of the stone in the cavity, the dissipation of the cavity, and so on. The results of the experiments and the theoretical premises arising therefrom would lead to far more general differential equations for the motion of the stone. These would require a more elaborate mathematical calculation to predict the path and motion of the stone and cavity. Sooner or later one would arrive at a theory for the behavior of the stone that when mathematically formulated, would predict reasonably well the behavior of the stone underwater.

Still further study would lead one into the questions of the effects of the shape and mass of the stone, the nature of its surfaces, and so on, each refinement leading to perhaps a more accurate mathematical model of the theory. If one wanted to control the flight of the stone, still more study would be required. Likewise, a theory predicting the entire path of the stone both in air and in water would lead one into still more problems.

The experiments and the comparison of the data would lead one into not only a dynamical approach to the theory and its formulation in, say, the language of differential equations, but would lead to the great importance of treating the motion as having a certain statistical distribution about a theoretical mean dynamical path. All of this would lead to more accurate description and theory of the underwater motion of the stone. The worth of any one version of the theory would be measured by the accuracy of prediction, by the order of its agreement with experiment, by its clarity, consistency, and perhaps sim-

plicity, by its validity over a wide variety of objects other than stones, and by its suggestive powers.

The problem of building the physical theory, using mathematical relationships and techniques as an aid and guide to both the formulation of the experiments and the theory, does not end here.

Further examination of the data and theory would reveal the need for a more thorough understanding of the formation and behavior of the cavitation that persists after the dissipation of the original entry cavity. Again more experiment, more theory, more mathematical models, predictions based on the models, comparison with experiment and theory, . . . , would be needed.

And to top off our problem we would still not have taken into full account the spread of the path of the stone through the air, the behavior of the stone as it enters the water, the effects of shape, entry angle, velocity, and the like.

It is this sort of thing that has been going on for years in attempts to understand and build a suitable theory for the movement of a body through water, to understand the formation and mechanism of cavitation around the body, and to learn of better methods for controlling the cavitation and the behavior of the body in air, at water entry, and in the water.

In all of the steps described above we see, hand-in-hand with the experiment, the procurement, the analysis, and the interpretation of the accuracy and meaning of the data, the formulation of a physical theory in mathematical language, the solutions of the equations elucidating the theory, the predictions of the behavior of the body and related phenomena by the solution of the equations with suitable, applicable boundary values, and the fit of these solutions with the data of experiment to determine the degree of confidence one can place upon the theories.

Thus we see in the formulation of such a theory certain steps which are common to the development of all sorts of physical and economic theories; and we see firsthand how mathematics may enter into the picture.

It is steps of this sort that occur so often and must be appreciated properly if one is to work successfully in a wide variety of the applications of mathematics to research and development work in industry and laboratories.

These steps may be described roughly as follows:

(1) An understanding and the power to predict is desired with respect to some class of physical phenomena (or economic situation).

(2) Experiments, observations, and data carefully obtained under controlled conditions are sought.

(3) From (2) an attempt to interpret the data is made, and a tentative theory of the phenomena stated.

(4) From the theory of (3) a mathematical model is constructed.

(5) From the mathematical model (4) solutions of the mathematical relationships involved are sought for cases of possible interest.

(6) The solutions found in (5) are interpreted in the physical model and in the real phenomena of interest.

(7) The faithfulness with which the interpreted results of (6) fit the observed phenomena is investigated. The result may reveal some measure of the adequacy of the theory (3) and its model (4). If the theory then is tested by comparing its predictions with the actual phenomena as observed in tests and the predictions prove to be good, the theory is taken to be a good one. If the predictions of the theory do not prove to be good upon comparison with tests, then the theory must be abandoned or revamped and the whole process as outlined repeated. In some cases there may be no suitable method for performing all of the tests necessary for confirming the theory, in which case the best obtainable experimental information and pertinent theoretical considerations are pooled together—some physical, some

mathematical—to yield what conclusions can be drawn as to the adequacy of the physical theory.

At every stage in the development of physical theories mathematics in some form appears: structurally, analytically, computationally, . . . , or otherwise.

A few examples to illustrate what I have in mind may be of interest.

A visit to the principal offices of any large aircraft company will reveal many different groups doing work which requires a good deal of mathematical skill. In the planning stages that precede any feasibility or design study leading to a new aircraft or engine one usually finds one or more groups deeply involved in the assessment of the proposed plans. Such groups examine the proposals prior to design from many points of view. They attempt to analyze and evaluate different possible versions of the general proposals. They study the proposals from the standpoint of their operational suitability, if built. They attempt to estimate the economic suitability of the proposals. And they examine the general feasibility of a proposal before any real detailed design is undertaken.

It has been found that mathematicians skilled in logical processes and well versed in aerodynamics problems are well adapted to enter this field.

The more expensive the proposal and the more difficult the financing of the project, the more important are such assessment groups. In both the United States and England we find such groups tied in with management at high levels. These groups are commonly known as Evaluation and Analysis Groups, Operations Analysis (Research) Groups, or Assessment Groups. Good assessment and good analytical people are very valuable because they have mastered the art of analyzing the proposals with great benefit to their companies. This often saves much needless expense and effort.

A visit to design divisions in an aircraft company will disclose the fact that hundreds of man-years are involved in the successful design of a great airplane. The costs and risks are so great and the tolerances, weights, and materials required are so critical, that a great deal of precise design calculation is absolutely necessary. The craft, as a structure, alone requires much stress analysis. All sorts of vibrational estimates must be considered. Automatic control systems, servomechanisms, and circuitry must be designed and the aerodynamic properties of the craft estimated as accurately as possible. Wind tunnel tests must be designed and performed. The engines must be studied carefully, and this involves many questions regarding heat transfer, thermodynamics, and the like. Each of these fields involves many branches of mathematics, such as Fourier analysis, matrix calculations, partial differential equations, non-linear equations, the mathematics of automatic control systems, elastic theory, ... In this kind of work both engineers and mathematicians are needed. They must work together. The engineer is too busy to master the necessary mathematics to the extent required; the mathematician is too busy to master all of the myriad of physical details. The calculations typically are so massive that large calculating engines are used.

A visit to any other great industry such as the heavy electrical companies, the electronics industry, the communications industry will reveal a similar use of mathematics in all sorts of settings. Browsing around a good technical library, paging through typical engineering, chemical, electrical, and physical journals will thoroughly arouse your interest and impress you with the extent to which mathematics is now routinely used.

The rate at which the new discoveries in physics and in materials are being translated into engineering development work is extremely rapid. Much mathematical work is involved, especially in those fields concerned with the development of new physical principles.

How mathematics as such arises

We have seen in the development of a physical theory how mathematics in various ways arises. When the mathematics so concerned is considered as an abstract subject in its own right, it might be said that one is dealing with pure mathematics. Much pure mathematics has arisen from a study of mathematics as such; and it is a matter of history that many theories developed strictly as mathematical science have later found great use in the development of physical sciences (e.g., Einstein in formulating his theories of relativity found already in existence a whole field of differential geometry thoroughly worked out which he could take over bodily; matric theory existed long before the electrical engineers found it of use in circuit theory).

It is significant that many developments in mathematics *per se* have received their impetus or stimulus in unsolved problems in the physical world. Thus, much of the mathematical research work going on today in the field of convex polyhedral sets and linear inequalities has its impetus in the development of what is called "linear programming," a subject which has grown out of a large class of logistic and economic problems of considerable importance.

Much of mathematics can be considered as a deductive system stemming from a suitable set of postulates P. From P, by processes considered as allowable in the mathematics concerned, various theorems T can be derived.

If P is so, then T is so.

In other words, the mathematical processes used "prove" that (if a certain logic is accepted) if the premises P are considered as true, then the conclusion T is considered to be true. Thus in a certain sense mathematics proves little about the "real world," and furnishes nothing about the "real truth" of P or T. Only if the premises P (and the logic used), when given a specific meaning in the field of application, have been demonstrated as consistent ("true") in the field of application, can one say that the conclusions T are consistent ("true") in the field of application. This limitation on the "truth" of a mathematical result should not be interpreted as underestimating the great importance of mathematics as such, as I have mentioned previously.

We have seen earlier that in the development of a physical theory (or theory of nature or an economic theory related to the real world) one builds certain premises based in part on the data of experiment, in part on earlier theories, and in part on some sort of human insight. And, with the aid of a mathematical model and the techniques of mathematics, one predicts from the selected premises P, and their anchorage in the field of application, certain results in the physical picture. The "truth" of the result T in the real physical world depends upon the actual experimental verification of the result; and a suitable verification then demonstrates the adequacy of the premises P. Once the physical theory has been demonstrated as well-founded and one accepts the truth of P, then one can consider that T is true. In the physical theory, mathematics demonstrates that if P is so, T is so. The prediction power of the theory stems from the anchorage of the premises in the field of application. The stronger the anchorage, the greater the predictive power of the theory.

But the fact that a principal function of mathematics in the field of application is in furnishing T, once a P has been chosen, should not be interpreted as detracting from the importance of the mathematics either as a subject in its own right or as a subject so necessary to progress in broader settings. Mathematical reasoning contributes fundamentally to the construction of physical theory. Mathematics contributes insight and understanding and supplies ways and means for "squeezing out" of the premises P all that is hidden

therein. It is a most indispensable aid and powerful tool in the fields of applications.

The deterministic versus the statistical approaches

Some of you are acquainted with the frequent sayings that "mathematics is an exact science," that "as long as the mathematics is O.K. the results cannot be wrong." How misleading such statements can be when they are misused. As we have just seen, it is seldom that the anchorage of the mathematics in the underlying field of application is exact, complete, or perfect; hence, the interpretations given to the mathematical results can seldom be considered exact in the field of application.

You are familiar with the so-called "exact" types of measurements, data, calculations, and predictions. In actual life many such quantities which one uses in mathematical or other considerations are inexact, even though we sometimes ignore the fact in our use of such quantities. In order to account for such inexactness and variations and to bring order out of this dispersion (and sometimes confusion) it is now commonplace in the applications of mathematics to bring into play the statistical and probability approaches, that is, the "measure theory" approaches of mathematics. Not only is the strictly deterministic ("exact," "dynamical") description used, but in addition the "gray area" of uncertainties in the field of application is systematically taken account of in a very orderly and logical manner. In the actual applications of mathematics to nearly all economic and physical fields, one takes thorough account of the vagaries of the data, as well as the imperfections of the mathematical model used in the field of application.

In other words, in the contemporary applications of mathematics the methods of mathematical statistics and probability theory play a very important role as major supplements to other forms of mathematical disciplines. It is with the aid of such technology that so many practical advances have been made in the art of predicting and controlling physical and economic phenomena. For example, two of the most elementary of such principles (statistical surveillance theory and statistical quality control) have contributed greatly to the improved production and reliability of almost every product now produced by mass production methods.

On the more advanced levels of statistical theory we find the notions of correlation theory playing a fundamental role in the development of improved control and communication equipment. In the process of gathering information by electrical means the incoming signal containing the desired information is usually distorted. When the distortion has random statistical features it is known as "noise." To remove the noise and recapture the desired intended message, some sort of electrical filtering is used. Various methods for doing this are known. The principles underlying the theory of time series and correlation theory have been taken over into the theory of filtering and have found wide uses. Electronic devices that in effect calculate correlation factors are now in use and play key parts in achieving the desired filtering.

Thus, in summary, we see that the applications of mathematics to all sorts of fields—industry, research and development laboratories, universities, . . . —are truly amazing and of great value indeed.

What does this mean in the world of education

Now what does all of this mean in terms of our schools, our educational policies, and our curricula offerings especially in the secondary schools.

There is every indication that the coming generations must be better equipped to handle the multitudinous technological problems of the picture. These generations must supply their own physicians, engineers, physicists, chemists, mathematicians, and so on. To do this properly, the young people in our schools who have the

necessary potential abilities must be adequately trained at a young age, so that as they come to maturity and find the fields in which they expect to make their careers they will be prepared to enter upon the necessary professional courses of study.

This means that for the good of all the people these children of potential ability should study mathematics, physics, chemistry, biology at an early age. And, somehow, such children should be placed in an atmosphere where they will want to do this irrespective of their ultimate careers. Such capable students should have completed calculus by the end of the twelfth year of school; likewise, by that time they should have completed what is now considered typical first-year college courses in physics and chemistry. That this can be done, and can be done without warping the students as real boys and girls, has been demonstrated. Surely, junior- and senior-high-school pupils who can pilot airplanes, rebuild cars, build television sets, and so on, can do this too, if they are steered in these directions.

But to achieve this result there must be a "will to do" on the part of the school authorities and the teachers. And such a program will help the other students who are endowed with those other abilities so necessary for the support of the more professionally gifted students. Such a program will be good for all.

And now, referring to the college and university training for the coming generations, I feel that such a program will permit our more capable young people to enter into their professional careers at a younger age than most of them now do, and at an age when they learn and progress more rapidly. By so doing, those of real ability will be able to be of greater and perhaps longer service to their fellowmen, whatever be their profession—physician, mathematician, physicist, engineer, ...

Now consider the fields of mathematics. A survey of activities in industry, in research and development laboratories, and in universities shows clearly that there is much room for mathematical experts whose work cuts across the fields of mathematics, physical science, engineering, and economics. This means that wherever possible our mathematics departments should train promising mathematical talent for careers in mathematics—not only for teaching and research—but also for nonacademic careers in industry and in research and development activities.

Conclusions

I have tried to tell you something about the way in which mathematics is used in the real world. I have sketched in a brief way the important role that mathematics *per se* plays in contemporary fields, the manner in which a physical theory is developed with the aid of mathematical methods, and I have mentioned a number of important current fields in which mathematics has found wide use. I have discussed a few current trends in mathematical applications and the impact on the world of education.

I hope that some of the things I have mentioned will prove useful to you in your teaching. I hope that something of what I have said will be useful to you in encouraging your potentially able students to follow a course of study which will prepare them adequately and at an early age for whatever type of career they may eventually elect. Especially do I hope that you will find it possible to steer students of potential ability through calculus and through traditional first-year college physics and chemistry by the end of their twelfth year of school, regardless of what is to be their ultimate career.

And, finally, I trust that you will continue to feel that there is much room for good mathematicians—especially those who cross their interests with the physical and economic sciences.

A "folded" one-track mathematics program

ROBERT C. MCLEAN, JR., *Washington High School, Los Angeles, California.*

Slide rule scales have been folded, so why not "fold" the mathematics curriculum?

As MORE AND MORE SUGGESTIONS for changing and adding to the high school mathematics curriculum are made, one becomes intrigued by the problem of organizing the material in some logical scheme. Quite a number of two-, three-, and multiple-track programs have been advanced. It is the purpose of this paper to present a new name and a new organization appropriate for some of these other new programs. Basically, it is proposed that the high school mathematics curriculum be organized as a six-year sequence of courses. To fit into the standard three- or four-year high school program, a six-year sequence must be "folded."

The courses would be organized as a single sequence with three important reference points inserted. Elementary courses: Math. 1, Math. 2, Math. 3, and Math. 4. Passing these courses or an equivalent arithmetic-aptitude test score would be required for high school graduation. Intermediate courses: Math. 5. Math 6, Math. 7, and Math. 8. Passing these courses would be required for general college recommendation. Advanced courses: Math. 9, Math. 10, Math. 11, and Math 12. Passing these courses would be required for engineering-physical science-mathematics college recommendation.

Original placement of the student in the sequence should be determined, of course, by the mathematical aptitude of the individual student. It is suggested that the irate parent who demands that his little darling be permitted to enroll at once in the college preparatory sequence be met with the calm reply that his child *is* in the college preparatory sequence—it may take just a little longer for him than for some other students.

Some schools may not have enough weak students to justify four semesters of elementary mathematics. Let them be so good as to preserve the numbering, anyway, even though their own courses may start with some number higher than 1. This would preserve the uniformity of the course numbers for all institutions using the system.

This purely mechanical device could have several advantages. It lends itself to the modern integrated courses. It should lead the curriculum makers to develop a further integration to include the lower (arithmetic) end, as well as the upper one, of the mathematics spectrum. Further, this plan should suggest that the steps between consecutive semesters should not be as great as the present ones between general mathematics and algebra and between algebra and geometry. For some slower student, it might offer an opportunity to "major" in mathematics (for high school graduation, of course). That is, he could start at an appropriate

level in the series and progress from semester to semester, if he is able, for the full period of his high school attendance.

As a purely mechanical device, of course, this plan can be applied in a less desirable manner. It could be used as a novel means of giving numbers to high school courses, just like the college courses. This is not the purpose of the suggestion. Its purpose is to try to pull together all high school mathematics courses into a sequential, integrated whole with adequate provision for individual goals at various levels.

Have you read?

BELDING, ROBERT E. "Today I Learned," *The Clearing House*, December 1955, vol. 30, pp. 225-228.

This little article is hard to classify. But it is well worth your reading if you have ever wondered what makes a good teacher. The author quotes items from his friend's diary. These will not only interest you; they will also cause you to think about your friends and yourself in relation to your profession. His quotes cover: contentment, personal relations with strangers and friends, students, past and present, students' contributions, and the surplus of teachers' talk. Read this, enjoy it, and see if it does not sum up for you, as it did for Mr. Belding, the characteristics of a good teacher.

BRY, ILSE, and DOE, JANET. "War and Men of Science," *Science*, vol. 122, November 1955, pp. 911-913.

This article should be read by all, in that it points out the changing role men of science have played in relation to war. Beginning in 1912 with the meeting of Scholars of Science and Philosophy on down to the 1950 Society for Social Responsibility of Science, we see a continually changing role played by the scientists. It is of interest to note that the late Einstein was active throughout this period and participated in both conferences.

The original spirit seemed to be total mastery of scientific fact with advances to abstractions and approaches to the philosophical. Then we see the unification of scientific ideas and such organizations as the Institute for Unity of Science. This historical review gives one faith in the future when he considers that in the last 40 years science has mastered ever more facts, but in the process has attained humility, simplicity, and a political consciousness mostly lacking in the past. The history of wars need not repeat itself.

CECIL, ANDREW ROCKOVER, "Education's New Frontier," *Association of American Colleges Bulletin*, December 1955, pp. 600-613.

Remember the first atomic explosion on July 16, 1945, and that 10 years later the first dinner was cooked using atomic energy? Your students will be interested in this article because of the summary of progress in the atomic world. You will be interested because of the implications for education. For example, your students will like to know that West Milltown, N. Y. is lighted with atomic power, that the University of Michigan has preserved food by radiation, that the yield of peanuts has been increased 30 per cent by use of radiation with fertilizer, that the United States has established the first school to train people from all countries in atomic energy for peace. In the field of automation they will be surprised to learn that 1,000 radios per day are made by 2 men and some machines, that 90 per cent of the light bulbs in the United States are made by 14 machines and 14 men, that vacuum tubes check your bank balance, make out your statement, and mail it to you.

This is the Second Industrial Revolution. Here is where the teacher comes in. Semiskilled workers must be raised to a higher level of work. For example, pipe fitters will become hydraulic engineers, electricians will become electronics men.

Schools must prepare more college-trained people. The executive must be well prepared because the change in business organization leaves little place for intuition. The last point will cause you to consider your objective in education. The author calls it "the supply of spiritual force." I am certain that this is a forward looking discussion that both you and your students will want to read.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

A problem on the cutting of squares

MATHEMATICS STAFF OF THE COLLEGE, *University of Chicago, Chicago, Illinois.*

An article with this same title appears in the April, 1956 issue of The Mathematics Student Journal. *The present paper supplies some general mathematical background to that* Student Journal *article, and continues it with a discussion of the converse of the problem treated there.*

INTRODUCTION

THERE IS an important theorem of geometry to the effect that *if two given polygons have equal areas, then either of them can be subdivided into a finite number of polygonal parts such that, upon rearrangement, these parts form the other given polygon* (i.e., form a polygon congruent to the other given polygon).

This theorem tells us that—without decreasing or increasing its area—a given polygon can be partitioned and reassembled into a right triangle, into a regular pentagon or hexagon, into one (or several) equilateral triangles, into one (or several) squares, etc. Thus the theorem gives rise to a general type of problem:

> *Given a polygon, to cut same into parts and rearrange these parts in such a way that there results another polygon (or polygons) of some specified sort.*

The mathematical character of such a problem becomes clearer when it is phrased in more precise geometric language: *Given a polygon, by geometric constructions to determine certain straight lines in the polygon and to prove that the parts into which these lines divide the polygon can be rearranged to form another polygon (or polygons) of some specified sort.*

Once the meaning of this general problem is clear, it is safe to adopt a more concise phrasing, viz., *Given a polygon, to transform it into another polygon (or polygons) of some specified sort.* The word "transform" used here is understood to have the sense "cut into parts, and rearrange same," hence, to imply that the area of the original polygon suffers neither increase nor decrease under the transformation.

Problems of this type were considered in antiquity, apparently by architects who designed the buildings and monuments of the ancient world. The practical methods these men devised to solve such problems were not always supported by proofs, and hence many of them were inexact. This last remark is borne out by Abul Wefa, a distinguished Arabian mathematician of the tenth century A.D.[1] At the beginning of his work *A book on geometric constructions* (fragments of which have come down to us in the form of notes by his pupils), Abul Wefa says: "In the following book we consider the decom-

[1] Abul Wefa (940–998) was a Persian astronomer whose life was passed in Baghdad. He was one of the last of the great Arabian scholars who translated and annotated Greek scientific texts. His principal work, besides *A book on geometric constructions*, was a translation-commentary of Diophantus. He also wrote on algebra and systematized the trigonometry known in his day. Nothing of his work survives, except by way of the writings of his pupils and successors. A translation (into German) of *A book on geometric constructions* appears on pages 318–359 of *Journal Asiatique*, 1855. For further information on this great mathematician, see M. Cantor, *Vorlesungen über Geschichte der Matematik*, 3rd ed. (1907), Vol. 1, pp. 742–748.

position of figures, a problem which is important to many technicians and which constitutes the subject-matter of their particular investigations. We confront problems of this type when we are required to decompose squares into smaller squares, or when we are required to compose out of several squares a single larger square. We shall present the basic principles of such problems. We emphasize that no method of solving one of these problems can be counted as exact and worthy of confidence if it is not founded on such principles; this applies in particular to methods presently used by the technicians." It should be added that certain problems of this type were correctly solved by the ancients (one such solution, by Abul Wefa himself, is presented below), and in more recent times other problems of this sort have been solved and some ancient solutions improved upon.

How can a systematic study of such problems be organized? One way is to fix upon a simple convenient polygon and to find methods by which it can be transformed into certain other polygons, and other polygons can be transformed into it. The polygon that naturally comes to mind for this key role is the square. Thus, if we can find a way of transforming a square into an equilateral triangle, and if we can find a way of transforming a pentagon into a square, then by combining these methods we have a way of transforming a pentagon into an equilateral triangle of the same area.

Let us assign to the square this central role. The original theorem then tells us that the square can be transformed into a right triangle, into a regular pentagon or hexagon, into one (or several) equilateral triangles, into two (or more) congruent smaller squares, etc. Further, the general problem previously derived from the theorem now takes a more special form: *By geometric constructions to determine certain straight lines in a square, and to prove that the parts into which these lines divide the square can be rearranged to form another polygon (or polygons) of some specified sort.*

In this paper we treat two particular cases of the general problem just stated:

Problem I. Given three congruent squares, to transform them into a single square (i.e., to cut them into parts and rearrange these parts in such a way that a single square results);

Problem II. Given a square, to transform it into three congruent squares (i.e., to cut the square into parts and rearrange these parts in such a way that three congruent squares result).

Problem I is the problem solved in the April 1956 issue of *The Mathematics Student Journal*. Problem II is evidently the converse of Problem I. We shall summarize the solution of Problem I from the *Student Journal*, and then give three solutions of Problem II. Before turning to our mathematical discussion, however, we interpolate some pedagogical remarks.

Some pedagogical remarks

As we have emphasized, "to transform" means "to cut into parts and rearrange same." Thus any problem of transforming a square into some other specified figure involves finding what cuts to make and finding what rearrangement of the resulting parts works. There is a sharp psychological difference between these two actions.

Given a square *already lined out,* to cut it and rearrange the parts into another figure is a matter somewhat on the order of a jig-saw puzzle. Here is one such "puzzle" which we suggest you work out before reading on.

Puzzle. On light cardboard draw a square $ABCD$ of side 10 (inches or centimeters). Now, following the labels of Figure 1, locate on the sides of your square the points L, G, N, R_1, and T_1 as follows: $AL=BG=7$, $AR_1=2.9$, $DN=4.2$, $CT_1=3.1$. Draw the lines AG and LN. On LN locate the point S_1 such that $LS_1=1.6$. On AG locate

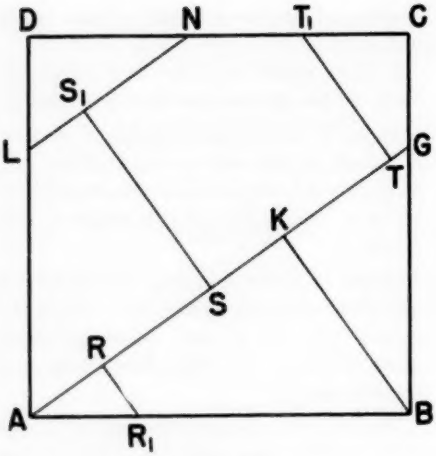


Figure 1

the points R, S, K, T as follows: $AR=2.4$, $RS=3.3$, $SK=2.4$, $KT=3.3$. Finally, draw the lines RR_1, SS_1, KB, and TT_1.

Cut out your square $ABCD$; then divide it into seven parts by cutting along the lines AG and LN, and along the lines SS_1 and TT_1, and RR_1 and KB.

Now rearrange the seven parts of $ABCD$ to form (a) a rectangle; (b) three squares of the same size; (c) a wide parallelogram; (d) a narrow parallelogram; and (e) an isosceles trapezoid.

But suppose, instead of giving the square already lined out (presenting Figure 1 and measurements which locate the lines in the square $ABCD$), we had simply said: In an unmarked square, construct lines and prove that the parts into which these lines divide the square can be rearranged to form a rectangle (or three congruent squares, etc.). The first reaction to this new *mathematical* problem is one of uncertainty: What lines should be constructed? What view of the problem can suggest even a start? How many parts should be aimed for? Clearly the passage from the puzzle (finding the arrangement) to the harder *mathematical* problem (finding both the cuts and the arrangement) imposes a radical change on the "mental set" of the solver.

Part of the explanation of this change surely lies in features of the problem, chief among which we consider its *indeterminateness.* The problem prescribes no pattern of cuts, and the number of resulting parts is not fixed in advance. A solution usually can be reached in several different ways (though a distinct effort is ordinarily required to find even one way to a solution), and the number of parts normally depends on the method of solution.

The indeterminateness of all these problems is one source of their pedagogical value: a student cannot progress to a solution without initiative, independence, and ingenuity. Another pedagogical value of these problems is the impetus they can give to the development of the student's geometrical intuition, and to the synthesis of his knowledge of separate geometric theorems and definitions and constructions (his knowledge *that*) into a unitary apparatus of ideas and methods (knowledge *what,* knowledge *how*).

Solution of Problem I

Let us return to Problem I, the problem of so cutting three congruent squares and rearranging the parts thereof that a single larger square results.

Suppose each of our three squares has area 1. Then, it might be thought, the problem is simply to construct a square with area 3, i.e., a square of side $\sqrt{3}$. This last is easy to do. Take one of our three given squares, label its vertices A, B, C, D, and note that the sides of this square are each 1 and its diagonal is $\sqrt{1^2+1^2}$, i.e., $\sqrt{2}$. Now refer to Figure 2 and proceed as follows: On DC extended, mark a point E such that DE has the same length as the diagonal AC, i.e., the length of DE is $\sqrt{2}$. Draw AE and note (since ADE is a right triangle) that its length is $\sqrt{1^2+(\sqrt{2})^2}$, i.e., $\sqrt{1+2}$ or $\sqrt{3}$.

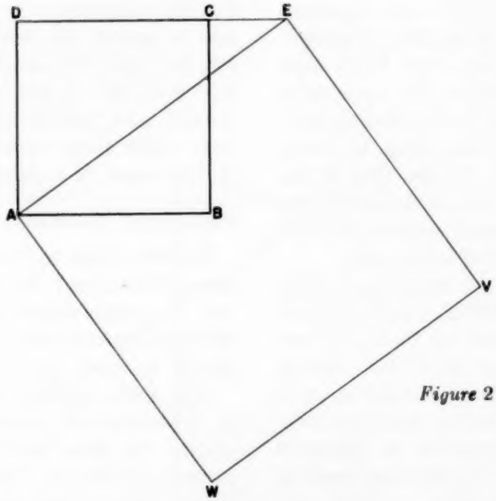


Figure 2

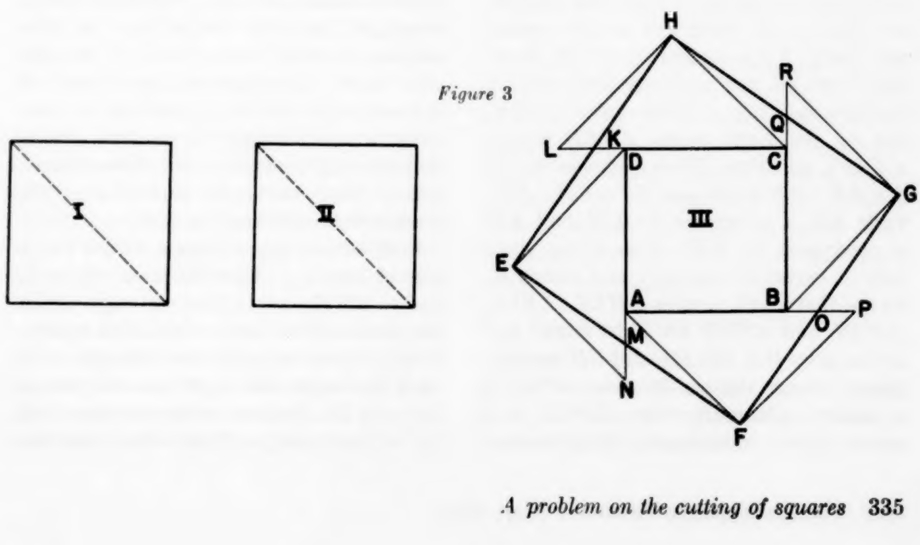
Segment AE having length $\sqrt{3}$, we construct a square with side AE and obtain the square $AWVE$ with area 3 as desired.

It is true that square $AWVE$ has area 3, and thus is a square composed in some fashion from our given three squares, each with area 1. However, square $AWVE$ is *not* a solution of Problem I: from the construction above we have no notion whatever of how to cut up our original three squares and rearrange their parts to obtain square $AWVE$.

Here is a genuine solution of Problem I. This solution is due to Abul Wefa, and is the one given in *The Mathematics Student Journal*.

Number our three given congruent squares I, II, III respectively. Cut squares I and II along a diagonal, and add the four resulting parts (they are congruent isosceles right triangles) to square III in the manner of Figure 3. Join the vertices E, F, G, and H with segments as shown. Cut the top right triangle along HK and fit the triangle HKL into the triangular space EKD; and do the same

Figure 3



along EM, FO, and GQ. The resulting quadrilateral $EFGH$ is the desired square.

Of this construction Abul Wefa says that it is "exact and at the same time suitable to the needs of the technician." To see that the construction is exact, we must prove that: (1) the area of the quadrilateral $EFGH$ equals the sum of the areas of the three given squares, and (2) the quadrilateral $EFGH$ is a square.

Let us prove (1) first. Referring to Figure 3 we see that $EFGH$ includes square III. Hence what must be shown is that the area of the part of $EFGH$ outside square III equals the sum of the areas of squares I and II. Notice that the little triangle HLK is congruent to the little triangle EKD, for the following reasons: HL is congruent to ED, $\angle HLK$ and $\angle EDK$ are each half a right angle and hence equal, and $\angle HKL$ equals $\angle EKD$. The same can be said about the next pair of little triangles, EMN and FMA, and about the remaining two pairs of such little triangles. Hence the area of the part of $EFGH$ outside square III equals the sum of the areas of squares I and II. Thus, finally, quadrilateral $EFGH$ has an area precisely equal to the sum of the areas of squares I, II, and III.

Now consider (2). To begin with, note that the small triangle EKD is congruent to triangle EMN, since: ED is congruent to EN, the two angles $\angle EDK$ and $\angle ENM$ are equal, each being half a right angle; and finally, KD is congruent to MN. From this result we can conclude that $\angle HEF$ *is a right angle* (for $\angle NEM$ equals $\angle DEK$, and so the right angle $\angle NED$ equals $\angle HEF$), and that EH *is congruent to* EF (for $EH = EK + KH$ and $EF = EM + MF$, while EK is congruent to EM and KH is congruent to MF). Proceeding similarly at vertex F, vertex G, and vertex H, we see that all the angles $\angle HEF$, $\angle EFG$, $\angle FGH$, and $\angle GHE$ are right angles and all the sides HE, EF, FG, and GH are congruent. Hence the quadrilateral $EFGH$ is a square. (Actually, that $EFGH$ is a square follows immediately by symmetry. For the construction is the same on each side of square III, hence the sides HE, EF, FG, and GH cannot differ in length nor can the angles $\angle HEF$, $\angle EFG$, $\angle FGH$, and $\angle GHE$ in which these sides meet differ from each other. Thus each of them must be a right angle.)

Problem II: preliminary consideration

We turn next to Problem II, the converse of Problem I: By straight cuts, to divide a given single square into parts which reassemble into three separate congruent squares.

Our first impulse regarding Problem II is somehow to construct in the given square the lines that appeared (in the square $EFGH$ of Figure 3) in Abul Wefa's solution of Problem I. This can be done, and we shall do so below as our first solution of Problem II. This solution sees our given square cut into *nine* parts.

On the other hand, the puzzle proposed earlier in this paper suggests that a solution of Problem II can be obtained by cutting our given square into only *seven* parts. We show how to effect this cutting in our second solution of Problem II.

And the matter does not stop there. In our third solution of Problem II we shall show how a cutting into *six* parts suffices.

The chief idea behind all three of these solutions is the following: If the three squares sought for are combined into a rectangle, then the longer side of this rectangle will be three times its shorter side. Hence an important aid to any of our solutions will be a method of constructing a rectangle whose area equals that of our given square, and whose longer side is three times the shorter one. We present that construction now.

Suppose our given square $ABCD$ has a side of length 1. Then its area will be 1, and so will the area of the rectangle which we propose to construct from this square. If the shorter side of this rectangle is x, then its longer side is $3x$ and its area is $3x \cdot x$, or $3x^2$. Since it is the case here that $3x^2 = 1$, we have for these sides x and $3x$:

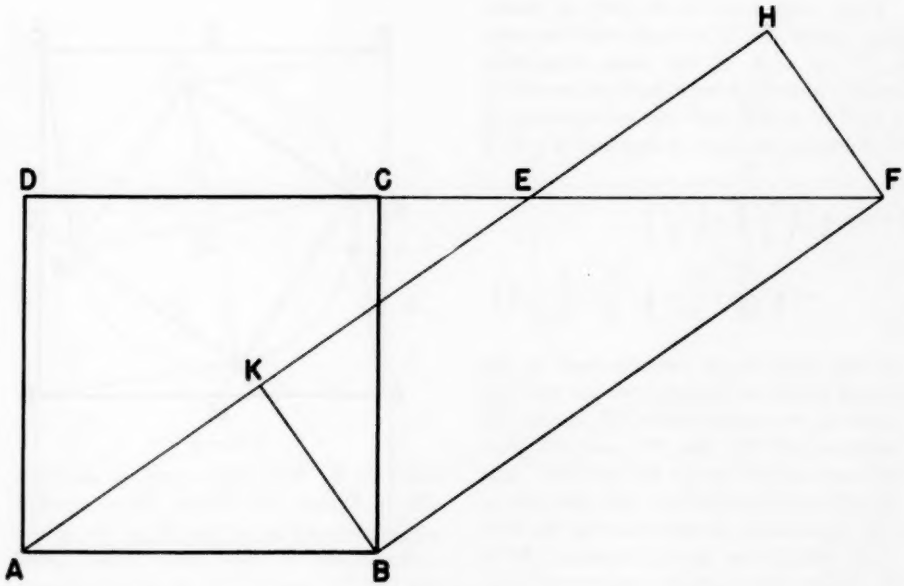


Figure 4

$$x = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

$$3x = \sqrt{3}.$$

The construction in Figure 2 has shown us how to obtain a segment of length $\sqrt{3}$; let us repeat that construction here. Referring to Figure 4, mark on DC extended a point E such that DE has length $\sqrt{2}$, the length of the diagonal AC. Draw AE and note that it has length $\sqrt{3}$, as we saw in connection with Figure 2. Now through vertex B draw a line parallel to AE, and let F be the intersection of this line with DC extended. The quadrilateral $ABFE$ is a parallelogram, since BF is parallel to AE and EF is parallel to AB. Further, parallelogram $ABFE$ has the same area, namely 1, as the square $ABCD$.

From B and from F draw the segments BK and FH perpendicular to AE extended. The resulting quadrilateral $BFHK$ is a rectangle whose area equals that of parallelogram $ABFE$ and hence that of our original square $ABCD$. Since triangle EFH is obviously congruent to triangle ABK (AB and EF have the same length, being opposite sides of a parallelogram; angles at K and H are right angles; angles at A and E are the same, since AB and DF are parallel) it follows that AE and KH are of the same length. Consequently: *rectangle $BFHK$ has area* 1, *a longer side of length $\sqrt{3}$, and a shorter side of length $\sqrt{3}/3$.*

We shall make use of rectangle $BFHK$ in each of the three solutions of Problem II that follow. We emphasize here, however, that the construction of rectangle $BFHK$ shown in Figure 4 is *not* a *transformation* of our square $ABCD$ into $BFHK$: Figure 4 does not show how $ABCD$ may be cut into parts that reassemble into rectangle $BFHK$.

First solution of Problem II

Let us solve Problem II by constructing in our square $ABCD$ the lines of Abul Wefa's solution of Problem I (the lines in square EFGH of Figure 3).

First, what facts do we have at hand? The square $ABCD$ we begin with has area 1. Hence each of the three congruent squares aimed for must have an area of $\frac{1}{3}$, a side of $\sqrt{3}/3$ and (in consequence of Pythagoras' theorem) a diagonal of $\sqrt{6}^3/3$

$$\left(\text{since } \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2+\left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}\right).$$

On the other hand, looking back at the square $EFGH$ of Figure 3 we see that the center of its inside square III is also the center of $EFGH$, that HC and DC have the same length (as do HL and DC), and that K is the midpoint of HE (and also of LD). And finally, in constructing the Abul Wefa lines in our present square $ABCD$, the square we construct corresponding to square III of Figure 3 must be given a side of $\sqrt{3}/3$ and must be so placed that each of its vertices is a distance $\sqrt{3}/3$ from some vertex of our square $ABCD$.

Together, these facts suggest the following construction for the Abul Wefa lines in our square $ABCD$: Draw a circle whose center is the center of $ABCD$ and whose diameter is $\sqrt{6}/3$ (the square we want inside $ABCD$ will be inscribed in this circle). Next, with D as center, draw a circle with radius $\sqrt{3}/3$ (either intersection of this circle with the previous circle can serve as a vertex of the desired smaller square inside $ABCD$). From here on the construction follows Figure 3 in a natural fashion.

We show the construction in Figure 5, and describe its successive steps as follows: Mark the center Z of our square $ABCD$ (Z is the intersection of the diagonals of $ABCD$). Now recall that in Figure 4 segment BK has length $\sqrt{3}/3$, and so segment AK in Figure 4 has length $\sqrt{6}/3$. Bisect the segment AK of Figure 4 (using a line through the midpoint of AB parallel to BK), and then with one of the halves of that AK as radius draw a circle cen-

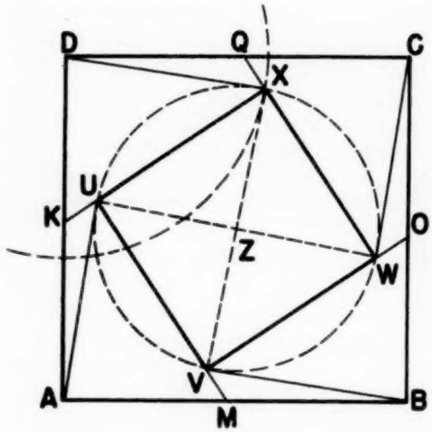


Figure 5

tered at Z. With the length of segment BK in Figure 4 as radius, draw another circle centered at D. Let X be one of the intersections of these two circles. Draw the line XZ and let V be its other intersection with the inside circle. Draw through Z a line perpendicular to XV, and let its intersections with the inside circle be U and W. With the help of the points X, U, V, and W we can now draw the Abul Wefa lines in square $ABCD$. Draw segment XU, and extend it to the side AD at K. Draw segment UA. Similarly, draw UV extended to AB at M, and VB. Again, draw VW extended to BC at O, and WC. And finally, draw WX extended to CD at Q, and XD. These four pairs of segments are the desired Abul Wefa lines in square $ABCD$. If square $ABCD$ is cut up along these segments, the resulting nine parts can be reassembled (in the fashion of Problem I) into three congruent squares each of area $\frac{1}{3}$.

We leave to the reader the proof that this construction actually transforms square $ABCD$ into three congruent squares. Hint: Draw a circle with K as center and KU as radius, and let L be the intersection of this circle and the extension of XK. Draw DL. Now use the facts of the construction to prove that K is the midpoint of AD and that XDL is an isosceles right triangle with side $\sqrt{3}/3$.

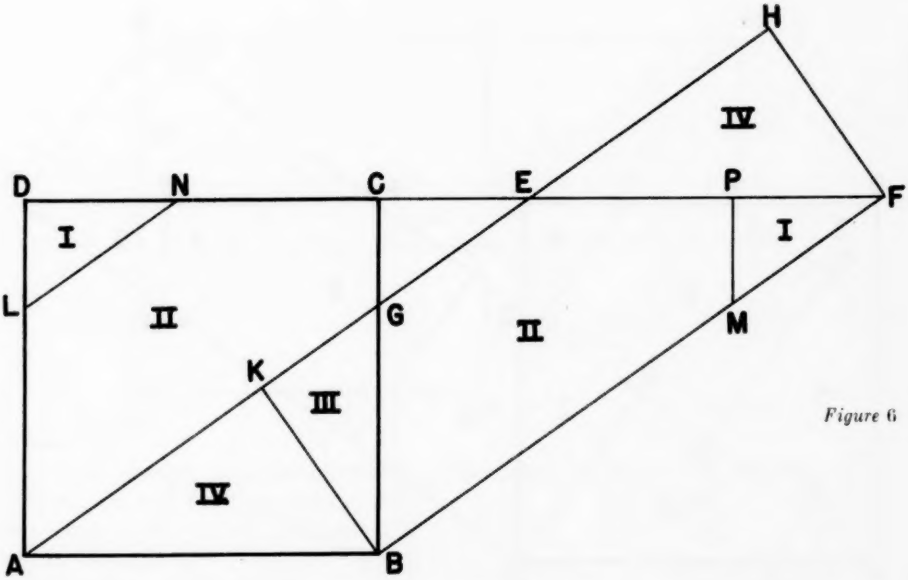


Figure 6

SECOND SOLUTION TO PROBLEM II

The solution now discussed will show how our square can be cut into *seven* parts which reassemble into three congruent squares.

Again, suppose our original square $ABCD$ has area 1. Make the construction of Figure 4 in $ABCD$, and let G be the intersection of AE with BC. On AD mark the point L such that AL and BG have equal length, and on BF the point M such that BM and AG have equal length. (Note that each of L, G, and M is equidistant from DF.) Draw from L a segment parallel to AE and ending at the point N on DC. Draw from M a segment perpendicular to EF and ending at the point P on EF. The segments AG, BK, and LN divide our square $ABCD$ into the four polygonal parts I, II, III, and IV shown in Figure 6; and segments EF, MP, and BG similarly divide rectangle $BFHK$ into the four parts shown in Figure 6.

We learned in connection with Figure 4 that square $ABCD$ and rectangle $BFHK$ have the same area. The construction of Figure 6 now enables us to show that $ABCD$ and $BFHK$ are each composed of parts which are pairwise congruent, i.e., we can now show how $ABCD$ *transforms* into $BFHK$. The proof runs as follows: Triangle BKG (viz., III) is common to both our square and our rectangle. Again, as we saw in connection with Figure 4, triangles ABK and EFH (viz., the two triangles IV) are congruent. Further, triangles LDN and MPF (viz., triangles I) is congruent because: segments LD and MP have the same length, and angles $\angle DLN$ and $\angle PMF$ are the same because their corresponding arms are parallel. And finally, in view of all these considerations, the pentagons $ALNCG$ and $BGEPM$ (viz., pentagons II) are evidently congruent.

Thus if our square $ABCD$ is cut along the segments LN, AG, and BK, the resulting parts can be reassembled into a rectangle.

The final stage of this discussion will show how to transform our square $ABCD$ into three congruent squares. We know that

$$(\text{length of } BK) = \frac{(\text{length of } BF)}{3} .$$

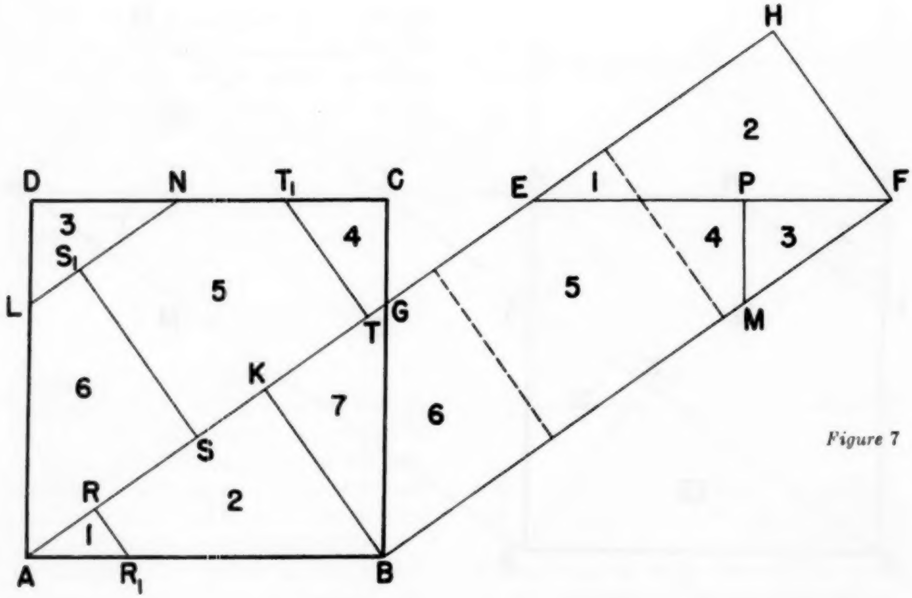


Figure 7

Divide BF into three congruent segments (lay off BK twice on BF, beginning at B), and at the two division points draw segments parallel to BK. These segments (they are shown as dotted lines in Figure 7) divide rectangle $BFHK$ into three congruent squares. Numbering the resulting seven parts of $BFHK$ as in Figure 7, let us look for congruent companions to these parts in our original square $ABCD$. On AG mark the points R, S, and T, such that each of segments KR, AS, and ST is congruent to BK. From R, S, and T, respectively, draw the segments RR_1, SS_1, and TT_1 perpendicular to AG. Let the resulting seven parts of square $ABCD$ be numbered as shown in Figure 7. Now it is easy to show that any two parts with the same number are congruent.

Thus if our original square $ABCD$ is cut along LN and AG, along SS_1 and TT_1, and along RR_1 and BK, the resulting *seven* parts can be reassembled into three congruent squares. We have concluded our second solution of Problem II, and incidentally resolved the "mystery" of the puzzle given earlier.

Third solution of Problem II

In this third and final solution of Problem II, we show how square $ABCD$ can be decomposed into three congruent squares by dividing $ABCD$ into only *six* parts. The key to this solution is to locate the rectangle $BFHK$ of Figure 4 in a different position on $ABCD$.

As usual, let our square $ABCD$ have area 1. According to the construction shown in Figure 4, the rectangle $BFHK$ then also has area 1, its longer side has length $\sqrt{3}$ and its shorter side has length $\sqrt{3}/3$. Using the symbol "(XY)" to designate the length of segment XY, the last two facts mentioned about rectangle $BFHK$ can be reported as follows:

$$(BF) = (HK) = \sqrt{3},$$

$$(BK) = (FH) = \frac{\sqrt{3}}{3}\,.$$

We now locate $BFHK$ in a different way on our square $ABCD$.

On DC extended mark a point W, such that $(DW) = (BF) = \sqrt{3}$ (segment BF is to be obtained from Figure 4).

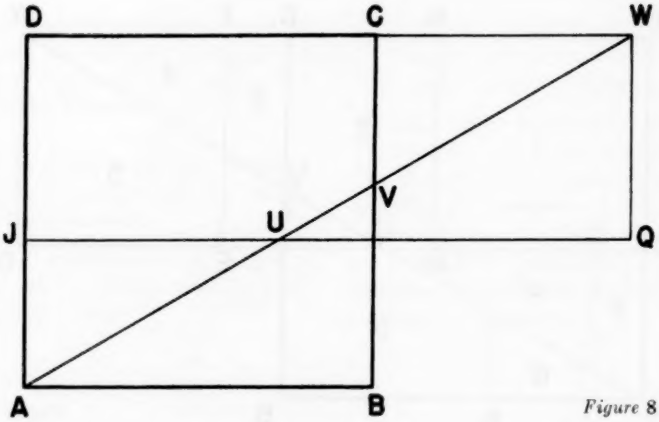


Figure 8

Draw AW, and let V be its intersection with BC. On AD mark the point J, such that $(DJ)=(BK)=\sqrt{3}/3$ (segment BK is to be obtained from Figure 4). Through J draw a line parallel to DW; let U be the intersection of this line with AW, and let Q be the intersection of this line with a perpendicular drawn from W. The rectangle $DJQW$ is evidently congruent to the rectangle $BFHK$ of Figure 4. Hence, as seen in Figure 8, we have located on $ABCD$ in a new position a rectangle of area 1 whose sides have the ratio $3:1$.[2]

Let us show that the square $ABCD$ and rectangle consist of parts which are pairwise congruent. The pentagon $DJUVC$ is common to both figures. It is sufficient, therefore, to prove that the triangles AJU and VCW are congruent, and that the triangles ABV and UQW are congruent. We know that

(1)
$$(CW)=(DW)-(DC)=\sqrt{3}-1,$$
$$(AJ)=(AD)-(DJ)=1-\frac{\sqrt{3}}{3}.$$

Further, since triangles AJU and ADW are clearly similar, it is the case that
$$\frac{(JU)}{(DW)}=\frac{(AJ)}{(AD)}.$$
In consequence of this equality and the facts of (1),

(2)
$$(JU)=\frac{(AJ)\cdot(DW)}{(AD)}=\frac{\left(1-\frac{\sqrt{3}}{3}\right)\cdot\sqrt{3}}{1}=\sqrt{3}-1.$$

From (1) and (2) it follows that $(CW)=(JU)$. Hence, the two similar triangles AJU and VCW are in fact congruent, and our proof is half done. From the congruence of AJU and VCW we see that
$$(VC)=(AJ)=1-\frac{\sqrt{3}}{3},$$
$$(BV)=(BC)-(VC)$$
$$=1-\left(1-\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{3},$$
and so

(3) $$(BV)=(QW).$$

Continuing, we have
$$(UQ)=(JQ)-(JU)=\sqrt{3}-(\sqrt{3}-1)=1,$$
and hence,

(4) $$(UQ)=(AB).$$

From (3) and (4) it follows that triangle

[2] The remarks of this paragraph incidentally provide us a simple way to trisect a right angle. Note that $\angle DAW$ is an angle of 60°, for
$$\tan(\angle DAW)=\frac{(DW)}{(AD)}=\frac{\sqrt{3}}{1}=\sqrt{3}.$$
Hence $\angle WAB$ is an angle of 30°. Using the familiar methods to bisect $\angle DAW$, we obtain the trisection of the right angle $\angle DAB$.

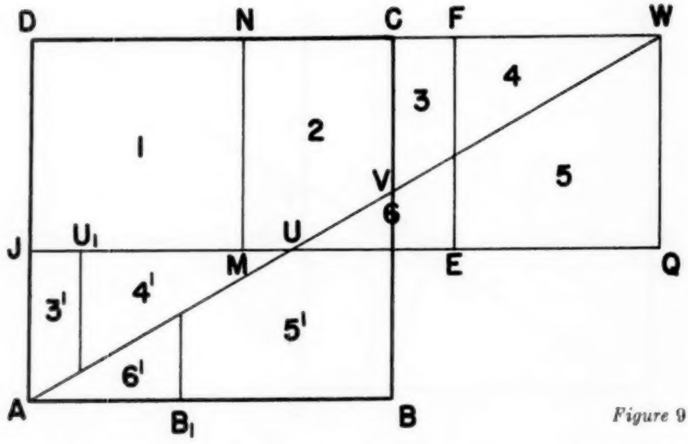


Figure 9

ABV is congruent to triangle UQW, and our proof is complete.

It is noteworthy that by cutting our square $ABCD$ along the *two* segments AV and JU we obtain *three* parts (they are the triangles AJU and ABV and the pentagon $DJUVC$) which reassemble into a rectangle. The method based on Figure 6 required three cuts and four parts.

As the final stage of our present solution, let us now subdivide rectangle $DJQW$ into three squares by segments EF and MN, and number the resulting parts of $DJQW$ as shown in Figure 9. It is not hard to find six corresponding congruent parts in our square $ABCD$. On JQ mark the point U_1 such that $(UU_1) = (WF)$; now, dropping a perpendicular segment from U_1 to AW, we divide triangle AJU into parts $3'$ and $4'$ congruent respectively to parts 3 and 4 of our rectangle. Again, on AB mark the point B_1 such that $(BB_1) = (WF)$; now, erecting a perpendicular segment from B_1 to AW, we divide triangle ABV into parts $5'$ and $6'$ congruent respectively to parts 5 and 6 of our rectangle.

Parts 1 and 2 being common to both our square $ABCD$ and the rectangle, it is now clear that the *six* parts 1, 2, $3'$, $4'$, $5'$, $6'$ of $ABCD$ can be reassembled into three congruent squares. Thus we have a solution of Problem II that requires cutting our square into only *six* parts.

Concluding remarks

It is seen from these three solutions of problem II that, in the course of 1000 years, the number of parts into which a square $ABCD$ may be cut to produce three congruent squares has been reduced from nine to six. Can this number be reduced still further? Is six the least number of parts that will work? At the present moment neither question has an answer. Moreover, both questions are probably hard to answer.

Another kind of question, related to our Problem II above, has recently received attention: Find constructions according to which a square can be partitioned into a finite number of smaller squares, *no two of which are congruent.* It has been proved that a square of side 175 can be divided into 24 squares whose sides are expressible in natural numbers, no two of these 24 squares being congruent; the question whether fewer than 24 such squares can be found is still open. There is a growing literature on this kind of problem.[3]

[3] Cf. Brooks, Smith, Stone, Tutte, "The dissection of rectangles into squares," *Duke Mathematical Journal*, vol. 7 (1940) pp. 312–340; Chowla, "Division of a rectangle into unequal squares," *The Mathematics Student Journal*, vol. 7 (1939) p. 69; Steinhaus, *Mathematical Snapshots* (New York: Oxford University Press, 1950), pp. 1–7. See also two articles by Goldberg in the *American Mathematical Monthly*, vol. 47 (1940), pp. 570–571, and in *Scripta Mathematica*, vol. 18 (1952), pp. 17–24, respectively.

Finally, here are three problems which can be solved by methods analogous to those of this paper, and on which you might like to try your hand.

A. Devise a construction by which an equilateral triangle can be transformed into a square.

B. Devise a construction by which a square can be transformed into an equilateral triangle.

C. Devise a construction by which a square can be transformed into two non-congruent smaller squares.

Many other elementary problems of this sort are treated in a recent little book, *The Wonders of the Square,* by B. Kordiemskii and N. Rusalev. Unfortunately this excellent book (on which parts of the present paper have been based) is in Russian and hence inaccessible to most American mathematics teachers.

Computer lops $29,999 off taxpayers' bill

Electronic computers used in engineering America's jet aircraft save the taxpayers many thousands of dollars every year.

As an example, mathematical problems which would cost $30,000 for every million operations by a clerk using a desk calculator are now done on these complex computers at a cost of only $3.00 per million operations. And, further, it is estimated that in the future, through wider uses of the electronic wonders, the cost for the same number of operations may be reduced to as little as 30 cents.

Not only behind the closed doors of aircraft engineers and scientists are these revolutionary new machines paying for their keep, but they have also added enormously to the operational ruggedness and reliability of modern supersonic military aircraft and commercial transports.

Devices aid other industry

The development of electronic units capable of withstanding extreme changes in temperature and severe pressures and shocks encountered at jet speeds and altitudes has also been essential in manufacturing the delicate electronic components. Ultra-premium tubes are capable of withstanding temperatures as high as 550° F.

Borrowing from the control principles used for aircraft and missiles, these electronics are now being used in many industrial processes.—*Taken from* Planes, *Official Publication of the Aircraft Industries Association of America, October 1955, vol. 11, no. 9*

No competition

Of the problems that confront United States aircraft engine designers and engineers, one of the most critical is weight.

It is easy to imagine the relief, then, of the design engineer of one aircraft-engine builder who on a recent vacation trip visited huge Boulder Dam in Arizona and observed a small metal plate attached to the lower section of a several-story-high turbine weighing hundreds of tons.

The plate was inscribed: "This turbine is not licensed for installation in aircraft."

Practical determination of the rank of a matrix

N. B. CONKWRIGHT, *University of Iowa, Iowa City, Iowa.*
Determining the rank of a matrix usually is not a problem for the high school. However, teachers may enjoy a simple method for finding answers to problems encountered in graduate courses.

Introduction

The procedure to be described for the determination of the rank of a matrix is similar to the evaluation of a determinant by the method of pivotal elements. The computation required is simple, and the method is very convenient in practice. Fundamentally, the theory involves only the familiar fact that a matrix is reducible by elementary row operations to a certain canonical form from which the rank is obvious. But as a systematic routine for actual numerical work, the method seems to be generally unfamiliar to students in spite of its useful nature.

The procedure has been discussed by methods different from those to be employed here in an article dealing with the solution of simultaneous linear equations by pivotal methods,[1] and even earlier in a paper by Wren[2] who also treated the material in the paper cited above.

Terminology

To facilitate the discussion the element in the upper left-hand corner of a matrix will be called the *leading element.* A matrix with leading element different from zero will be called a *standard matrix.*

We now define what is meant by a *pivotal reduction of a matrix.* Let P be any non-zero matrix having at least two rows and at least two columns. Denote the number of rows and columns of P by m and n, respectively. If the leading element is zero, let P be transformed by the interchange of rows and/or columns into a matrix Q which has the leading element different from zero. Denote the element in the jth row and the kth column of the standard matrix P or Q (as the case may be) by x_{jk}. Then compute the elements of the $(m-1)\times(n-1)$ matrix R having for the element in the jth row and the kth column the quantity

$$\begin{vmatrix} x_{11} & x_{1,k+1} \\ x_{j+1,1} & x_{j+1,k+1} \end{vmatrix}.$$

The procedure by which matrix R was obtained from the given matrix, including (if P had a zero leading element) the transformation of P into a standard matrix, is called a pivotal reduction of P.

Determination of the rank of a matrix

Let A_1 be a non-zero matrix (having more than one row and more than one

[1] N. B. Conkwright and J. D. Heide, "The Solution of Simultaneous Linear Equations," The Mathematics Teacher, XXXVIII (April 1945), 177–80.

[2] F. L. Wren, "Neo-Sylvester Contractions and the Solution of Systems of Linear Equations," *Bulletin of the American Mathematical Society,* XLIII (December 1937), 823–34.

column) whose rank is to be found. The procedure for determination of rank is as follows. First, effect a pivotal reduction of A_1, and thus obtain a matrix A_2. If $A_2 \neq 0$, and has at least two rows and two columns, carry out a pivotal reduction of A_2, thus obtaining matrix A_3. If $A_3 \neq 0$ and has at least two rows and two columns, effect a pivotal reduction of A_3, and thus obtain a matrix A_4, and so on. If this pivotal reduction is repeated a sufficient number of times, it is obvious that we shall finally obtain either a matrix of one row, a matrix of one column, or a zero matrix. *If A_h is the last non-zero matrix which can be obtained in the manner described, then the rank of A_1 is h.* A proof of this statement is presented below.

In order to simplify the computation, it is permissible to simplify any matrix A_s by performing transformations which do not change the rank, before carrying out the pivotal reduction which leads to the next matrix A_{s+1}.

An example

Let it be required to find the rank of the matrix

$$A_1 = \begin{pmatrix} 3 & 1 & 4 & 0 & 2 \\ 1 & 0 & 2 & 3 & 5 \\ 0 & 3 & 4 & 6 & 1 \\ 2 & 4 & 6 & 3 & -2 \\ 0 & 3 & 1 & 2 & 3 \end{pmatrix}.$$

Then

$$A_2 = \begin{pmatrix} -1 & 2 & 9 & 13 \\ 9 & 12 & 18 & 3 \\ 10 & 10 & 9 & -10 \\ 9 & 3 & 6 & 9 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -30 & -99 & -120 \\ -30 & -99 & -120 \\ -21 & -87 & -126 \end{pmatrix} \sim \begin{pmatrix} 3 & 4 & -2 \\ 0 & 0 & 0 \\ 7 & 29 & 42 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 \\ 59 & 140 \end{pmatrix}.$$

Hence the given matrix is of rank 4.

Theoretical discussion

To establish the validity of the test for rank it will first be shown that the rank of A_2 is less by one than the rank of A_1. Assume for the present that A_1 has a leading element different from zero, and consider the partitioned matrices

$$A_1 = \left(\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}\right)$$

$$B = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 \cdots & 0 \\ \hline -a_{21} & a_{11} & 0 & 0 \cdots & 0 \\ -a_{31} & 0 & a_{11} & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_{m1} & 0 & 0 & 0 \cdots & a_{11} \end{array}\right)$$

where it is understood that B is an m by m matrix. Then it is found upon multiplying the partitioned matrices above that

$$BA_1 = \left(\begin{array}{c|c} a_{11} & a_{12} \; a_{13} \cdots a_{1n} \\ \hline 0 & \\ 0 & \\ \cdot & A_2 \\ \cdot & \\ 0 & \end{array}\right)$$

Now B is non-singular, since it was assumed that $a_{11} \neq 0$. Hence A_1 and BA_1 have the same rank. It is therefore obvious that the rank of A_2 is less by one than the rank of A_1.

Let it be recalled that A_h denotes the last non-zero matrix obtainable from A_1 by repeated pivotal reduction. Hence A_h has only one row or only one column, or else pivotal reduction of A_h results in a zero matrix A_{h+1}. But the rank of A_h is one. (If A_h has more than one row or more than one column, an argument like that at the beginning of this section, coupled with the fact that $A_{h+1}=0$, shows the rank of A_h to be one.) Now a repetition of earlier arguments reveals that for $s=2, 3, \cdots,$

h, the rank of A_s is less by one than the rank of A_{s-1}. Therefore, if r denotes the rank of A_1, the rank of A_h is $r-h+1$. Hence

$$r-h+1=1, \quad \text{and} \quad r=h.$$

The test for rank has thus been established.

If the leading element of A_1 is zero, the above argument may be applied to the standard matrix obtained from A_1 by interchange of rows and/or columns.

Some special cases

Pivotal reduction is especially easy for square matrices of certain special types. Thus, if a standard matrix Q is symmetric, the matrix R obtained by pivotal reduction is also symmetric. Hence, all elements on one side of the principal diagonal of R can be written without any computation. The reader may find it interesting to investigate the form of R when Q is Hermetian, skew-Hermetian, or a circulant matrix.

Letter to the editor

Q.E.D.—Another Russian Invention?

Dear Sir,

In his article "*Quod erat demonstrandum*" in the January 1956 issue of The Mathematics Teacher, Harold P. Fawcett pleads for the introduction of the inventive spirit into the geometry classroom. To this, no alert mathematics teacher can take exception. But the attempt to link the Q.E.D. teaching practice with thought control and to insist that its very spirit is antidemocratic is nonsense, and dangerous nonsense at that.

The case against Q.E.D. is concentrated chiefly in this paragraph: " . . . An even more serious question . . . can be raised concerning the cumulative effect on a student who is faced with data day after day, either given or assumed, and precisely what conclusion he must derive from these data or just what it is he must prove. Does not such a practice violate the very spirit of scientific method? Where in the world is this practice found except in the demonstrative geometry classroom and in countries where thought control is common? . . . Is there any respectable scientific investigator anywhere who will support the practice of making prejudgments or of determining in advance the conclusions to which inquiry must lead? . . . "

Of the three questions raised by the author the second is semantically charged and consequently difficult to answer in a straightforward manner. If the author means that the pupil in the geometry classroom is forced to accept the stated conclusion even if logically unjustified, I deny that the pedagogic practice he describes implies such a compulsion. (And his earlier tribute to Miss Agnes Murphy would tend to support this denial.) The practice of stating the given and the conclusion is no doubt found in the demonstrative geometry classrooms of countries with general thought control as well as in the classrooms of those countries that abhor thought control. For centuries the practice persisted in England with no adverse effect on her democratic institutions. In our own country the practice does not seem to have had any important influence on political institutions. This type of geometry teaching poses no subversive threat to any type of government; arguments for changing it had better come from other fields of human activity.

In the remaining two questions the author asks: Do scientific investigators make prejudgments? The answer is yes; they have to. We are dealing here, not with thought control, but with experimental control. In most cases the investigator has a pretty good idea of the outcome of an experiment, but he checks carefully the anticipated with the experimental results. A prejudgment may or may not be experimentally verified. But the investigator is free to perform any experiment (not destructive of life or property) suggested by his inquiring mind.

In the opinion of the author "such experience . . . frequently encourages intellectual dishonesty. . . . His [Girolamo Saccheri's] erroneous belief that he had actually proved the parallel postulate is a good illustration of what can happen when one is under the compulsion of proving a predetermined conclusion." This is an interesting thesis, and I do not know how widely mathematicians subscribe to it. But certainly the same compulsion existed for Gauss, for Bolyai, for Lobachevsky, and for Riemann, none of whom can be accused of intellectual corruption. The extensive efforts made to prove the predetermined conclusion that $x^n+y^n=z^n$ is not solvable in integers for integral $n>2$ led to no intellectual dishonesty (of which I am aware), but to much useful mathematics.

However, we are not here concerned with scientific induction; the chief objective in the teaching of geometry is to establish the nature of mathematical proof. (Witness the *Fifteenth*

Continued on page 392

The supervisor plans a program through self-evaluation

LESTA HOEL, *Portland Public Schools, Portland, Oregon.*
Self-evaluation procedures have often proved to be of considerable value. In this article, self-evaluative criteria are developed for the supervisor of mathematics.

Evaluation is the process of making judgments that are to be used as a basis for planning. It consists of establishing goals, collecting evidence concerning growth or lack of growth toward goals, making judgments about the evidence, and revising procedures and goals in light of the judgments. It is a procedure for improving the product, the process, and even the goals themselves.[1]

LEARNING is evaluated in terms of changed behavior on the part of the learner. A supervisory program is successful only insofar as it results in the changed behavior of all those people who work directly or indirectly with children, resulting ultimately in the improvement of the learning situation for them. The learning process for a school staff begins by establishing goals and attempting to attain these goals. They must constantly evaluate the results in terms of the changed learning situation for the children, and set up modified goals for further learning.

A self-evaluation program for a supervisor consists in: (1) setting up the goals (changed behavior) which, in his opinion, a school staff should strive to attain, (2) developing and applying criteria for measuring his share in the achievement of these goals; (3) planning new goals and procedures in light of the apparent results. These three procedures are illustrated in this paper. "Staff" means the teachers and administrative personnel within any school unit. The evaluation criteria are stated for a city-wide supervisor of mathematics whose relationship to the principal and teachers in the schools is that of an advisory capacity only. Similar criteria could be set up for anyone in a supervisory position.

THE FOUR GENERAL OBJECTIVES ASSUMED FOR AN ADEQUATE SCHOOL PROGRAM

I. The staff improves the learning situation for children through an adequate program of self-improvement.
II. The staff works co-operatively on a program of child growth and development.
III. The staff prepares and selects adequate materials of instruction and guides.
IV. The staff works with the community in planning the school curriculum

[1] Kimball Wiles, *Supervision for Better Schools*, (New York: Prentice Hall, Inc., 1950).

SPECIFIC OBJECTIVES AND EVALUATION CRITERIA BASED UPON THESE OBJECTIVES

I. The staff improves the learning situation for children through an adequate program of self-improvement.

Objectives of school staff	*Evaluation criteria for city-wide supervisor*
The staff:	
1. makes a critical analysis and clarification of their philosophy of education	1a. Am I acquainted with the philosophy of the district and with this philosophy as it operates within the individual school? b. Have I tried to express my own philosophy as it applies to education in general and to mathematics? c. How does my philosophy compare with the philosophy of the district? What are the similarities? What are the differences?
2. plans a long range in-service training program	2a. Have teachers had a share in planning the in-service program set up for the district? b. Have I sought help in the in-service training of teachers of mathematics other than regular classes (1) from teachers? (2) from colleges? (3) from the community? c. Have I a long range in-service training program planned which will be available to teachers of mathematics?
3. studies and evaluates techniques of instruction such as	
a. democratic procedures in the classroom	3a. Have I given help to teachers in the use of teacher-pupil planning?
b. use of available resources	b. (1) Is there a list of available resources in mathematics? (2) Is there evidence of the use of concrete materials in teaching mathematics?
c. attention to individual differences	c. What are the evidences that more attention is being paid to individual differences than five years ago?
d. evaluation of results in order to determine progress of each child	d. What are the evidences that pupils are learning more mathematics now than five years ago?
4. participates in meetings such as	
a. regularly scheduled in-service classes	4a. (1) Have I used teachers in the system as resource people in these classes? (2) Have they been well attended?
b. general meetings of (1) teachers of high school mathematics (2) teachers on grade levels	b. (1) How many general meetings have I called this year? (2) Were they optional? (3) How well attended were they? (4) Was opportunity given for teachers to ask questions? (5) Did they share experiences and materials? (6) Were the meetings teacher planned? (7) Were these meetings evaluated by those who attended? (8) Were the reactions from these meetings favorable? (9) Did I get and use suggestions for improvement of the meetings?

c. professional faculty meetings	c. (1) In approximately what per cent of the faculty meetings attended was I aware of the problems of the school? (2) Was the discussion centered around vital problems suggested by the teachers? (3) Have I made any follow-up of these meetings?
d. conferences	d. (1) Have individual conferences generally been suggested by the teacher? (2) Have I had favorable reactions from these? (3) Have I asked for and used suggestions for improving these?
5. plans demonstrations and inter-visitations.	5a. Did my demonstrations grow out of the teacher's needs? b. Did I prevent a feeling on the part of the children that the teacher was inadequate? c. (1) Have I planned inter-visitations? (2) Was it preceded and followed by a conference? (3) Has it resulted in better classroom situations? d. Have I asked for and used suggestions for improvement?
6. uses outside consultants	6a. Have my services been used to: (1) help evaluate a program to determine the problem? (2) help limit the problems? (3) introduce the problem as an outsider? (4) provide additional resources? (5) provide specific information? (6) measure progress? (7) help evaluate work that has been accomplished? (8) help determine next step? b. Do all principals ask freely for my assistance? c. Have I asked for and used suggestions for improvement of my service as a consultant?
7. orients new staff members	7a. Have I assisted in the orientation program? b. Have I had definite plans for assisting new staff members? c. Have I eliminated the obstacles which I have encountered in assisting these new people? d. Have I asked the new staff to suggest ways of orienting the next year's teachers?
8. studies how experienced teachers profit from their day-to-day experiences	8a. Do I build a teacher upon his strengths? b. Do I help inexperienced teachers to profit from the success of the experienced person? c. Can I discover strengths of teachers and share them with others?
9. studies and practices group dynamics	9a. Do I relinquish the leadership to others within the group? b. Do I allow the group affected by a decision to help make it? c. Am I a good member of a group in a non-leader role?
10. constantly evaluates their program of self-improvement	10a. Have I been asked to assist in establishing criteria for judging the improvement of instruction? b. What tangible evidence is there that the staff is evaluating its program?

11. makes a critical analysis of individual characteristics which make for success in the positions which they hold	11. Have I helped the staff to improve in the following characteristics? a. Pleasant reaction to unpleasant situations. b. Co-operation with the school and the community. c. Attention to physical appearance and comfort in the room. d. Appearance. e. Professional attitude.

II. The staff works cooperatively on a program of child growth and development.

Objectives of school staff	*Evaluation criteria for city-wide supervisor*
The staff:	
1. improves the physical setup	1. What evidence is there in the rooms that mathematics is a live subject?
2. is willing to try new procedures	2a. Have I helped the staff to experiment? b. Have I helped the staff to evaluate results of these experiments? c. How many of these: (1) have resulted in improved procedures? (2) have been discontinued?
3. provides adequate diagnostic and remedial measures in the skills	3a. Have I given assistance in the use of tests? b. Have I given assistance in diagnostic and remedial work? c. Have I helped the staff become more resourceful in preparation and use of diagnosis?
4. evaluates pupil growth	4a. Is the staff becoming more interested in the child than in the mathematics which they are teaching? b. What concrete evidence is there to show that pupil growth has been evaluated? c. Have I helped the staff to improve its procedures for evaluating growth?
5. knows how we learn and applies this knowledge in learning situations within the school	5a. Do I apply the laws of learning as I work with teachers as an example to them? (1) Have I helped the teachers set up goals for learning situations? (2) Do I recognize the teacher's goal? (3) Do I realize that the teacher's goal may not be the goal which I have for that teacher? (4) Do I encourage teachers to experiment? (5) Do teachers know that I will recognize an apparently unsuccessful experiment as a step in the learning process? (6) Do I help the teacher to recognize success and establish new goals? b. Have I helped the staff know and apply laws of learning in relation to pupils, just as I use it in relation to teachers?
6. recognizes and provides for all children	6. What provisions have I helped teachers make for a. the slow learner? b. the gifted? c. the emotionally disturbed child?

III. The staff prepares and selects adequate materials of instruction and guides.

Objectives of school staff	*Evaluation criteria for city-wide supervisor*
The staff:	
1. takes an inventory of needs	1a. Do I know the needs of teachers for materials in general and in specific buildings? b. Have I helped the staff become skilled in the selection of materials?
2. selects criteria for textbook adoptions	2a. Have the criteria set up for textbook adoption taken into account recent research? b. Have I helped the staff set up and apply these criteria to the selection of books?
3. prepares and uses materials: a. units	3a. Are the units the result of actual classroom experiences?
b. bulletins	b. Do these serve a definite purpose?
c. guides	c. (1) Is it short? (2) Is it definite? (3) Was it planned co-operatively? (4) Are helps easily found?
d. materials for classroom use	d. Is opportunity provided for teachers to construct classroom materials?
e. bibliographies	e. (1) Are they available to teachers? (2) Are resources mentioned available.
4. continually examines and evaluates new teaching materials as they appear	4a. Am I acquainted with recent materials and research? b. Do I publicize new materials? c. Have I stimulated the staff to search for new materials? d. Have I set up with them criteria for critical evaluation of materials?
5. becomes familiar with materials already available	5a. Have I made an inventory of the use of available materials of instruction? b. Have I publicized such material? c. Have I encouraged the staff to use them? d. Have I been asked frequently to assist with the use of new materials?

IV. The staff works with the community in planning the school curriculum.

Objectives of school staff	*Evaluation criteria for city-wide supervisor*
The staff:	
1. plans the use of community resources	1a. Have I made available community resources in mathematics? b. Have I helped the staff to plan the use of community resources?
2. conducts study groups with members of the community	2a. Have I been invited to participate in community study groups? b. Have these been constructive rather than of a defensive nature? c. Have I helped the staff plan study groups? d. Have I urged teacher use in study groups?
3. Gets consensus of school and community in establishing goals, formulating procedures, and evaluating progress	3a. Have I used the community on mathematics committees? b. Have I helped the staff work with the community in formulating procedures? c. Have I helped the staff make the community aware of constant evaluation of progress? d. Have I encouraged the use of pupils in these activities?
4. Maintains membership in civic organizations	4a. Do I have affiliation with some civic groups and meet and recognize members of the staff there? b. Is it active participation or merely membership? c. Have I helped teachers find their place in civic life?

A six-year program of supervision based upon the evidence collected in the evaluation

I. As it relates to in-service training of school personnel:

A. *Teachers*

1. Plan a long range in-service training program in co-operation with teachers and principals.
2. Use outstanding teachers as leaders of in-service classes.
3. Recognize and publicize the strengths of teachers.
4. Plan more inter-visitations, particularly on the high school level.
5. Hold area meetings as demand warrants.
6. Make more opportunities for casual visits to establish contacts with teachers and principals who do not welcome supervision by a supervisor not directly connected with the individual school.
7. Bring more teacher problems and teacher participation into faculty meetings.
8. Improve lines of communication between supervisor, principal, and teacher.
9. Use influence where possible to improve teacher training on a statewide basis.
10. Co-operate with the national association to improve the teaching of mathematics.

B. *Myself*

1. Be more aware of opportunities to use the group process and of the roles of the members of the group.
2. Keep abreast with research in education and mathematics.
3. Make a continued effort to see the teacher's viewpoint.
4. Improve the following personal characteristics which will make my work more effective:
   a. Be a "we," not an "I," person
   b. Co-operate with the other administrators
   c. Give praise where deserved
   d. Admit errors
   e. Be punctual
   f. Make judgments only after knowing all the facts
   g. Be well groomed and appropriately dressed
   h. Cultivate outside interests
5. Ask for and use suggestions from teachers and other administrators.

II. As it relates to the child:

A. Experiment further with an arithmetic clinic; train and encourage teachers to use this approach in a remedial program.
B. Plan with teachers a program for the prevention of emotional difficulties arising from arithmetic.
C. Plan with teachers a program for the gifted in mathematics.
D. Encourage arithmetic near graduation on the high school level, both remedial and consumer.
E. Co-operate with the guidance department in making pupils aware of mathematics needs and capabilities.

III. As it relates to materials:

A. *For the classroom*

1. Enlarge the mathematics laboratory where teachers may make materials for classroom use.
2. Evaluate commercial materials and make them available.
3. Encourage the establishment of an arithmetic corner in every room, including measuring instruments, visual aids, and games.
4. Build up a mathematical classroom library.
5. Encourage the use of materials for pupils of varying abilities in each classroom

B. *For the teacher*

1. Encourage professional libraries.
2. Issue bulletins regularly.
3. Encourage and plan for interchange of materials and ideas among teachers.
4. Issue a primary arithmetic guide.
5. Co-operate with the state in preparing a high school guide.

IV. As it relates to public relations:

A. Encourage study groups with parents.
B. Hold parent conferences when requested to discuss any of the child's mathematical problems.
C. Co-operate with professional people, particularly engineers and scientists in encouraging and training adequate personnel for the professions.

Our national debt reduced to 54,253,475 dominions?

HENRY CLARENCE CHURCHMAN, *Council Bluffs, Iowa.*

A steak dinner costing one gold donion? Is it possible? While gold donions and dominions (monetary) may be for future generations, this article does provide an interesting idea for our generation to dream about.

RECENTLY there has appeared in the public press a considered statement that it is not illegal to have in your possession in the United States of America a twenty-, ten-, five-dollar, or any other denomination U.S. gold coin.

As a lawyer I must qualify an approval of that statement with an affirmative only under certain conditions. It is not legal tender. And you may not carry it in your pocket as spending money. But if you can qualify under the classification of numismatist, you may hold it and expose it to admirers. And you may transfer title to another coin collector, in which event you may lawfully exact of him as many as thirty-six paper or silver dollars, U.S. legal tender, for a twenty-dollar gold coin. This is the going price if the coin is in fine condition. Wear, tear, or blemish will reduce the cost to you and your take from another collector.

The press release, it is possible, was in response to a growing public interest in the return of the United States of America to the gold standard. Actual value of a coin, as of a good diamond, is said to create in its possessor greater confidence, élan, aplomb, a sense of stability. Some people think we are substituting for gold the higher quality automobiles in America. The circulation of gold coins as legal tender is lawful in Canada, in England, in France, and almost everywhere except in the United States.

Possibly the people of the United States of America are turning again to a desire to think in terms of intrinsic values, to treasure the feel of a gold coin in the pocket.

If that be true, the U.S. Treasury and the Congress of the United States are faced with a dilemma.

With the official price of gold pegged at approximately $35.00 an ounce (480 grains), it is not logical to refer to an ounce of gold as a "twenty-dollar gold piece." Practically all of our gold coins now have been refined by our U.S. mints, converted into gold bullion, and most of it stored at Fort Knox, Kentucky.

It is equally illogical, decimally speaking, to designate the ounce, half-ounce, and quarter-ounce gold coins as "thirty-five," "seventeen-fifty," and "eight-seventy-five" pieces.

But there is an out—a possible solution. The duodecimalists of the world could solve the problem easily. And their method is not one bit mysterious. Basically, they favor counting everything, so far as practicable, by dozens. If we, and Canada, and Great Britain are of one mind to establish an international monetary system of uniform intrinsic values and denominations, yet each with its own insignia and country of origin stamped upon it, the possibility of a new standardized currency, of uniform legal terms and values common to all, is present now.

New money pegged	*Numerical contents pegged*		*Pegged English money*	*Pegged Canadian, U.S. money*
One Dominion	(dozen doz. doz. doz. Nions)	P	20,736 shillings	$4,976.64
Half Dominion	(six dozen dozen dozen Nions)	P	10,368 shillings	2,488.32
Quarter Dominion	(three dozen dozen dozen Nions)	P	5,184 shillings	1,244.16
One Minion	(one dozen dozen dozen Nions)	P	1,728 shillings	414.72
Half Minion	(six dozen dozen Nions)	P	864 shillings	207.36
Quarter Minion	(three dozen dozen Nions)	P	432 shillings	103.68
One Renion	(one dozen dozen Nions)	P & G	144 shillings	34.56
Half Renion	(six dozen Nions)	P & G	72 shillings	17.28
Quarter Renion	(three dozen Nions)	P & G	36 shillings	8.64
One Donion	(one dozen Nions)	P & G	12 shillings	2.88
Half Donion	(six Nions)	P & S	6 shillings	1.44
Quarter Donion	(three Nions)	S	3 shillings	.72
One Nion	(one Nion)	S	1 shilling pegged at	.24
Half Nion	(six Edonions)	S	6 pence	.12
Quarter Nion	(three Edonions)	S	3 pence	.06
One Edonion	(one Edonion)	C	1 pence	.02
Half Edonion	(six Erenions)	C	Half-penny	.01
Quarter Edonion	(three Erenions)	A	One farthing	
One Erenion	(one Erenion)	A		

P indicates paper money; *G* denotes gold coins; *S*, silver coins; and *C*, copper coins. *A* indicates subdivisions for accounting purposes, tax levies, etc. The Erenion and the Quarter Edonion (as well as the farthing and the U.S. mill) have no existence as coins.

Furthermore, any one of these sovereign nations could act separately and establish by law its own fixed exchange values, or any two could act in concert. The United States and Canada are present examples of uniform monetary terms and values, each nation acting independently. If the new monetary system be established, the terms "dollar" and "sovereign," for a period of time used concurrently with the new currency, will eventually disappear. Numismatists, as a group, give evidence of the impermanence of tens and hundreds of once-popular coins.

Can you imagine our referring to the U.S. national debt of approximately $270,000,000,000.00 as 54,253,475 *dominions*? (See the table for value of one proposed dominion.)

A General Assembly of the state of Iowa—I cite from a local source where I claim some small knowledge—when the millage levy had risen from some sixty mills on one thousand dollars of "taxable value" to an imposition of over two hundred mills, quietly changed the method of evaluation of property so as to let the new millage levy read something less than sixty mills on the same property.[1] The taxpayer, relieved by the reappearance of a low millage levy, continued to pay the same amount of tax dollars; and the new millage continued its historical, nation-wide climb toward bigger and better tax dollars.

[1] In Iowa, it had been directed by law prior to 1933 that the assessor should depreciate the actual value of a forty-acre tract of land to one-fourth its actual value (and upon this valuation the levy of approximately 200 mills was applied). By change in the law, the assessor was directed to omit the step of dividing the actual valuation by four. This had the effect of quadrupling the assessed value of property, enabling local tax levying boards to reduce their millage levy by approximately 75 per cent. Its effect was, like a local anaesthetic, to deaden the pain while continuing a necessary operation.

Prior to 1933 (Sec. 7109, Code of Iowa 1924, 1927, 1931), the local assessor was told that "All property subject to taxation shall be valued at the actual value which shall be entered opposite each item, and, except as otherwise provided, shall be assessed at twenty-five per cent of such actual value.

By Act of the Forty-fifth General Assembly of Iowa (1933), Chapter 121, Sec. 75, the local assessor was redirected that "All property subject to taxation shall be assessed at its actual value which shall be entered opposite each item. The terms 'actual value,' 'assessed value' and 'taxable value' shall hereafter be construed as referring to 'actual value.' The tax rate shall be applied to the actual value, except as otherwise provided."

In like manner with the U.S. national debt, there is some relief in being able once more to confront it in terms of millions.

Canadian, United States and English small coins contrasted

In England, from time immemorial (probably extending back to the Roman occupation), a dozen pence have equalled the value of one shilling (24¢ in U.S. or Canadian exchange up to 1914). This has permitted the English to mint half-shillings and half-pence. It is also possible to divide a shilling or a penny by four and get a whole number, free of any fractions. Thus, four farthings equal a penny in England, and one-fourth of a shilling equals three pence.

In the U.S. and Canada, you cannot take one-half or one-fourth of the quarter-dollar and avoid a fraction. If two articles sell for a quarter-dollar, one should sell for 12½¢. We do not coin that fraction; so the purchaser pays 13¢.

You cannot divide a nickel in two, or 15¢ in two.

England, in the area of our quarter-dollar, unquestionably has a more satisfactory system than do the U.S. and Canada with their cents, nickels, dimes, and quarters. England is not likely to abandon these advantages.

There have been some pressures for the minting of a U.S. 7½¢ or 2½¢ piece. But either promises to furnish only partial relief and has been shunted aside time and time again.

Now, it is possible, if a new currency were set up on the dozenal base, that many untold wonders might be enjoyed in daily trading, in the exchange of minor coins, in the levying of sales taxes—all in a field where the small purchaser, the first to suffer and the least able to bear it, ekes a living. It is, I believe, the half-cent of tribute, unavoidable as it now is, that cuts deeper into the American common mind than a dollar paid for tobacco.

Mechanics of erecting the system

In England, one possible procedure is to keep the shilling, and peg it at 24¢ as legal tender in England, Canada, and the United States of America. Great Britain, to do this, might find it necessary to peg the pound sterling at $4.80 in U.S. and Canadian money.

Again, the English Parliament might, by statute, peg the pound sterling at $2.88 U.S. and Canadian dollars, and provide that it shall equal the value of 12 "Nions," of 144 "Edonions," or of 1728 "Erenions."

A more practical solution is to coin the new gold Donion at $2.88, containing approximately 40 grains of gold, and establish it as the value of 12 Nions, of 144 Edonions, or of 1728 Erenions.

Think of the fractions to be avoided in England, in Canada, and in the United States. And think of the easing of demand for 1¢ coins in the U.S., resulting from the increase of small items to 6¢, 12¢, and 18¢, if merchants find it unnecessary to give 2¢, 3¢, or 4¢ in change. Moreover, the 24¢ item, plus 1¢ sales tax, might be paid with our existing quarter-dollar, at least during the changeover.

The ease with which the price of one item may be determined, when its value by the dozen or the gross is known, should not be overlooked.

But if the shilling in England be pegged at 24¢, all coins of lesser value would continue their relationship to the shilling, and continue to be minted and circulated with no thought of eventual retirement. However, all coins and currency in England above the shilling, in any event, would be created in new denominations; and in Canada and the U.S. all coins and currency would be in new denominations.

The table of proposed paper money and gold, silver, and copper coins indicates their pegged value in U.S. and Canadian currency. For a dozen years, both new and old currency might be printed and coined. Their related values indicate how easy it would be to pay in either or a combination of both.

Effective mathematics in industry[1]

W. S. BAUMGARTNER, *North American Aviation, Inc., Downey, California.*

With the existing shortages of technical manpower, it is important to know what mathematics the industrial community wants the schools to teach.

It has been my privilege to attend, as a representative of industry, several recent meetings with high-school and junior-college mathematics teachers. The purpose of these meetings was to discuss the educational requirements of employees in modern industry, particularly with reference to the student whose education, for one reason or another, must end at these levels. I believe everyone who attended these meetings is convinced that the type of employee industry needs has changed markedly in recent years and will continue to change in the foreseeable future. An acute and increasing shortage exists in the supply of people with adequate educational backgrounds for the type of work we now require, while fewer and fewer jobs are available for the unskilled person. Company educational programs can help with specific training, but for necessary mathematics and basic science we must depend on our high schools. I hope to outline some of the most frequently needed—and often missing—mathematics.

Let me make clear at this point that I most certainly recommend that whenever possible the student should complete college. In most branches of engineering we have five jobs per graduate and the situation is worsening. However, not all students find it possible to complete college. Some find it financially impossible, and some are not well adapted to education at this level. We do not need college trained engineers for all our technical work. A ratio of five technicians to one engineer is considered desirable, and there is no doubt that many companies find it difficult to make maximum utilization of scarce engineers due to lack of technician support. Splendid opportunities exist in industry for the man without a degree, if he possesses a certain minimum education. We need machinists, sheet metal layout men, tool makers, laboratory assistants, electronic technicians, draftsmen, and engineering assistants. In any one of these classifications the door to advancement is open to any man who has the qualifications.

One thing I would like to get across to your students—our future employees—is the direct relation between their preparation and their future rate of pay. Anything that affects the pocketbook interests us all and is easy to put in terms that we can understand. Without necessary educational background—and mathematics is the basis of this—it is impossible to progress to the jobs that mean more pay, more security, and less hard uninteresting work. I frequently talk with people who, after starting their work career, realize their deficiencies and bemoan the fact that they did not get the education when they had an easy opportunity. Some of these people are able to get ahead by self-instruction or night school, but all will agree that this is the hard way and that they will suffer considerable delay in their

[1] Address given at the California Conference for Teachers of Mathematics, July, 1955.

careers. I think any one of them could do an excellent job of convincing today's high-school student of the fallacy of following the easiest course.

The reasons for increasing educational requirements for industrial employees will be apparent to anyone who considers the changing type of products of our factories. Where machine tolerances of .001 inch were once considered close, tolerances of .0001 inch are now common. Angles are measured in seconds. Surface finishes with root mean square roughness tolerances of a few micro-inches are commonplace. Automatic machine control by electronics is in regular use. Complicated electronics systems requiring testing and maintenance are major industrial products. Our supply of skilled people must keep pace with our technical advancement.

Before attempting to list the most needed shop mathematics, I talked with many men in all branches of manufacturing. These men represented various levels of supervision in shops doing sheet metal work, precision machine work, assembly, electronics, design and engineering. I hope by this means to have these recommendations represent a cross-section of industry rather than any specialized field. I was surprised by the simple and basic nature of their requests. In fact, I feel I must apologize for the low academic level of my recommendations. Anyone expecting a discussion of the need for matrix algebra or LaPlace transforms is due for a disappointment.

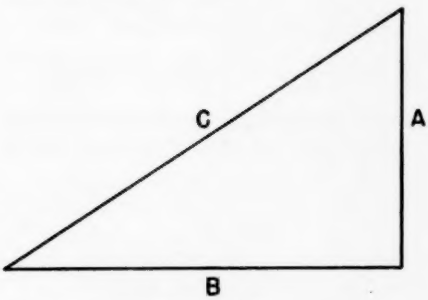
The following, not necessarily in the order of importance, are some of the most commonly mentioned deficiencies in mathematics:

1. *Change fractions to decimals*

   When this was first mentioned to me I thought the man was joking, but as it was brought up by almost every shop supervisor interviewed it became apparent that a real problem exists. The superintendent of a large shop told of a young man who was sent to a sheet metal stock bin to get a piece of $\frac{1}{8}''$ thick aluminum. After a long time he returned to report that there was no $\frac{1}{8}''$ aluminum in the bin, but there was a sheet marked .125 that looked to be almost an eighth and he wondered if that would do. Not only should they be able to change fractions to decimals by the process of division, but they should be able to think with ease in either system. We cannot post enough conversion tables to have one visible from all locations!

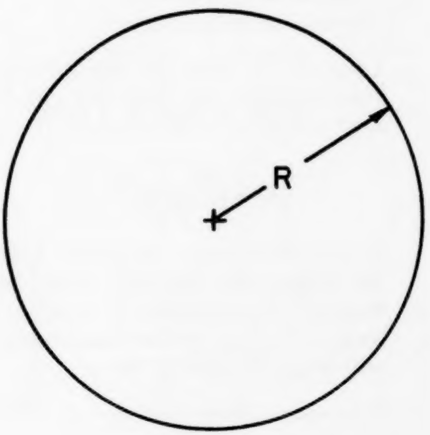
2. *Major geometric relations*

   Such formulae as



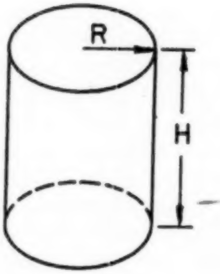
$C^2 = A^2 + B^2$

Figure 1



$A = \pi R^2$

Figure 2



$V = \pi R^2 h$

Figure 3

and others are in such common use as to require some knowledge. The exact formula needed will depend to some extent on the type of work.

3. *Isolate any unknown in an equation*

As an example, consider the familiar equation for resonance in a tuned circuit:

$$2\pi FL = \frac{1}{2\pi FC}$$

where:

F = frequency in cycles per second
L = inductance in Henries
C = capacitance in Farads

I have seen a man with this formula before him and the values of C and L known, unable to convert it to

$$F = \sqrt{\frac{1}{4\pi^2 CL}}$$

in order to solve for frequency. I do not suggest that the man be able to handle difficult or involved equations, only simple ones (such as the example) which are in everyday use.

4. *Solve simple linear equations*

Again I do not suggest the solution of difficult or involved equations, only simple ones the use of which fall within the scope of a technician's work. A practical example is the selection of the correct wattage rating for a resistor in an electrical circuit. The wiring diagram may show a 50,000-ohm resistor across a 200-volt circuit. In the stockroom there is a choice of 50,000-ohm resistors with ratings ranging from $\frac{1}{4}$-watt to 100-watt, size from $3/16'' \times \frac{1}{2}''$ to $1'' \times 8''$, and cost from \$.10 to \$4.00. I have known an assembler faced with this problem who, although familiar with the expressions $E = RI$ and $P = EI$, was unable to make the simple substitution to get the equation $P = E^2/R$ which he could then solve to find that his resistor must handle 4/5 of a watt, and the 1-watt size would be his best choice.

5. *Understand what a logarithm is and how it may be used*

I have never seen an occasion in shop work where a logarithm table was actually used in the solution of a problem. However, a knowledge of logarithms is necessary in order to understand the operation of the slide rule and other analog computing devices and to grasp the significance of the decibel so commonly used in sound and electrical work.

6. *Use of the slide rule*

The accuracy of the slide rule is adequate for the major portion of our work, and it is a universally used instrument. Its speed makes its use mandatory in many operations. Almost every ambitious man sooner or later learns to use it by night classes, self-instruction, or impromptu classes conducted by other employees. It is not necessary to learn all of the multiple operations and short cuts possible on the more complex rules, but it would be highly desirable for a person to do multiplication, division, powers, and roots.

7. *Understand and use basic plane trigonometry*

There is no responsible job in any shop that does not require some trigonometry. It is a "must" in any layout or drafting job; machinists use it in precision measurement; and the electronics technician finds it indispensable in understanding alternating current circuits. A complete course in plane trigonometry may be beyond practical limits, but surely the high-school graduate should be familiar with the functions of angles and how trigonometric tables may be used. In the case

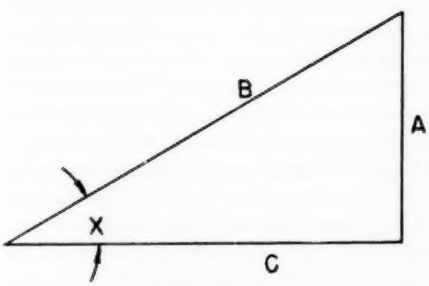


Figure 4

he should know that:

$$\sin X = \frac{A}{B} \qquad \cos X = \frac{C}{B}$$

$$\tan X = \frac{A}{C} = \frac{\sin X}{\cos X} \text{ etc.}$$

The ability to use these simple relations will be invaluable.

8. *Basic vector algebra*

As an extension of the study of trigonometry, an understanding of the solution of problems in vector algebra is desirable. Problems involving stresses in any framed structure require an understanding of vectors and the voltage, and current relations in alternating current circuits can be explained only by their use. Sufficient familiarity with the subject to understand the relations of the voltages E_O, E_L, E_C, and E_R, shown on the schematic diagram and the vector diagram, will be adequate (Fig. 5).

9. *Graphical analysis and presentation of data*

More and more data and statistical information is being recorded or presented graphically. This is done to clearly and quickly show trends and because complex data, if presented numerically, would be too cumbersome. Much data is now automatically recorded graphically as a plot of a variable against time or one variable against another. The oscilloscope, so widely used in analytical work, is a common form of graphical presentation. The ability to think in terms of graphs as well as prepare them is desirable.

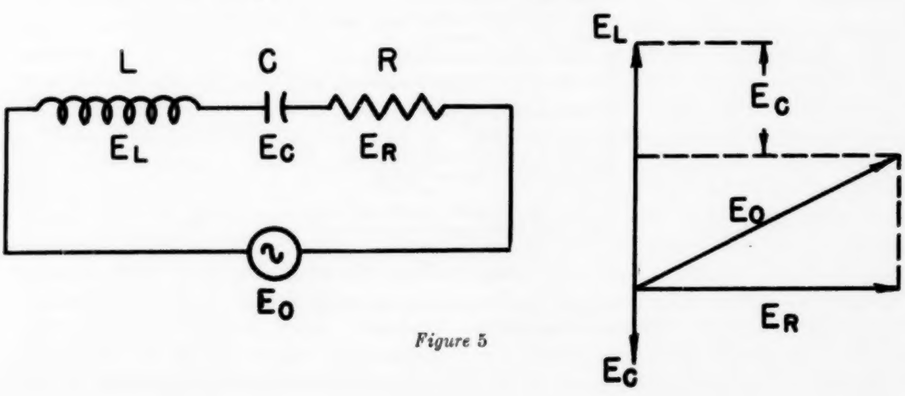


Figure 5

10. *Use of negative and positive powers of 10*

This is a simple and very useful tool to avoid decimal point errors in complex solutions. Unfortunately, it is seldom used. Substituting in an equation, we may come up with an expression

$$X=\frac{(.000369)\ (8.27)\ (10{,}000)}{(.0000014)\ (3.44)}.$$

Solution of this expression as it is invites decimal point error. However, if the expression is rewritten as

$$X=\frac{(3.69)\ (10^{-4})\ (8.27)\ (10^{4})}{(1.4)\ (10^{-6})\ (3.44)}$$

$$=\frac{(3.69)\ (8.27)\ (10^{6})}{(1.4)\ (3.44)}$$

it is easily solved with small chance for error.

The preceding recommendations by no means constitute a mathematical education; they are merely common weaknesses noted in many high-school graduates. In general, industry would prefer people thoroughly grounded in and able to use the more ordinary kinds of mathematics, to those with a poor and unusable knowledge in a wider field. Certainly any steps that can be taken to improve the students' facility in using common mathematics will be appreciated by our supervisors and will materially improve the employees' chances for advancement.

I find from the mathematics teachers that a major problem exists in convincing the student of the desirability of securing an adequate mathematics background. Admittedly the mathematics course is more difficult and requires more study than many others, and there is a tendency to elect the easier courses and make only a minimum effort in the mathematics that is required. I think this is a common problem in human nature. We are willing to put forth effort only if we feel sure of an adequate return. Perhaps our problem is to show the student that he will receive an adequate return for his effort. To this end it has been a pleasure to take part in discussions with many teachers, and if we have made even a little progress, I am sure it has been worthwhile.

Algebra

What does Algebra mean?
Add or subtract to find someone's scheme.
I have y and I have x
Now by golly, what comes next?
This is what Algebra means.

I struggle and fight
Stay up all night.
Finally I get my answer right.
This is what Algebra means.

Upon this thought, sweet and pure,
You may often and long endure:
For without Algebra you see,
Next year there will be no Plane G.

—*Jacqueline Yon, Altoona Senior High School, Altoona, Pennsylvania.*

Desirable alterations in order and emphasis of certain topics in algebra

RALPH BEATLEY, *Harvard University, Cambridge, Massachusetts.*

The 1950's are transitive years for high-school and college mathematics. Here are some suggestions for desirable changes in the present courses.

AT TIMES ONE is asked, "Will modern mathematics effect changes in high-school algebra?" Taken literally, this invites one to guess what the future will bring forth. I believe that the real intent of such questions is to ask, "What desirable alterations in order and emphasis of the topics of high-school algebra are suggested by recent trends in college mathematics?" It is this subject that I propose to consider.

First of all, what are these trends? As I look back over the last 30 years, I see the gradual disappearance of college algebra from Freshman mathematics and the introduction of more and more calculus in the Freshman year. Concomitantly, I note the intrusion of secondary-school algebra into the Freshman year, often under the label "College Algebra." If this is a modern trend, it is also a return to the custom of 150 years ago. I hope that this is only a temporary aberration, soon to be corrected by better educational theory and practice at the secondary level. In any case it does not concern this discussion, so I will not revert to it. I see a marked increase in instruction in statistics, and, in the last few years, the offering of instruction in statistics to Freshmen as alternative to some of the calculus of the Freshman year, with the aim of serving those whose interest is primarily in the social studies rather than in the physical sciences. I see the supplanting of instruction in "the theory of equations" by instruction in abstract algebra or "higher algebra," which is "higher" than "college algebra" and is concerned in part with the logical structure of algebras.

Having noted these trends, I am ready now to consider what they suggest with respect to high-school algebra. In doing so, I shall have in mind every pupil who can fairly be expected to derive some benefit from the study of algebra beyond the first half-year of instruction in that subject, regardless of his or her ultimate enrollment in a college. This first half-year of algebra, sixteen weeks or so, is ordinarily devoted to formulas, graphs, equations, and problems, all with unsigned numbers; negative numbers and operations on signed numbers; equations, graphs of equations, and problems the solution of which may involve signed numbers. I would leave all this as it is.

When we come to the second half-year of algebra, however, which is ordinarily devoted to factors, fractions, fractional equations, equations with literal coefficients, simultaneous equations of the first degree, radicals, and the first steps in solving quadratic equations, there is, as I see it, a conflict of interests. There is little in these topics that I have just listed that touches closely the lives of the pupils who study them. This objection is not enough, in my opinion, to condemn these topics as inappropriate for the second half-year of algebra. Yet I wish that they had more immediate appeal for the pupils. In the case of those who will go on to college and

become interested in science or in the social studies, a thorough grounding in the fundamentals of elementary algebra is absolutely necessary. Even so, I think it is possible that mastery of these fundamentals can be won in a significantly shorter time than my colleagues and I devoted to them when I was teaching secondary-school mathematics. In any case, I should prefer to insert ahead of the usual topics of the second half-year whatever important topics in algebra I can find that seem to touch the pupils' lives more closely. I want material that is of intrinsic interest to the pupil, or of interest because it is immediately useful to him. I hope in this way to increase the appeal and the value of algebra at this level, not only to bright pupils and to average pupils, but also to some of the faithful who are slightly below average in intellectual power.

I have never forgotten a remark by David Eugene Smith that, from the point of view of difficulty, there is no reason why the arithmètic and geometric progressions should not appear much earlier in elementary algebra. I suppose it is our preoccupation with equations in the body of elementary algebra that has relegated the progressions to a sort of appendix to elementary algebra, along with the binomial theorem. To my mind, and I believe to many pupils also, the progressions are interesting for their own sake because of the number relations inherent in them. I note that the progressions often appear in general intelligence tests where no special knowledge of mathematics is presupposed.

Closely related to the progressions are the three averages—the arithmètic mean, the geometric mean, and the harmonic mean. Most pupils assert with confidence that an automobile that goes at the rate of 20 miles per hour for one hour and then at the rate of 30 miles per hour for one hour has an average speed of 25 miles per hour for the two hours. At this point they feel no need to contemplate the real meaning of the average of several numbers as a single number that, when substituted for each one of the several, will produce the same result. They either pick the number midway between 20 and 30, or add 20 and 30 and divide by 2. But when confronted with the problem of finding the average speed of an automobile that goes uphill at the rate of 20 miles per hour for 1 mile and then downhill at the rate of 30 miles per hour for 1 mile, they are baffled unless they go back to first principles and divide the total distance by the total time in order to find the average speed. Their answer, 24 miles per hour, can be shown to be the harmonic mean of 20 and 30, the reciprocal of the arithmètic mean of the reciprocals of the given rates.

A third problem, that of finding the average rate of increase of a population over a span of three decades which increases by 4 per cent in the first decade, by 6 per cent in the second decade, and by 3 per cent in the third decade, is of still a different sort and yet fundamentally the same in that it seeks a number that, when substituted for 1.04, 1.06, 1.03, will increase the population by the same amount in three decades. This number is the cube root of the product of 1.04, 1.06, 1.03; that is, it is the geometric mean of these three numbers. These averages involve ideas that are obviously useful; and the content of two of my examples is close to the pupils' lives. When considered in greater detail than I can go into here, these averages also involve algebra. Further, there is a symmetry between the algebra of the arithmètic mean $(a+b+c)/3$ and of the geometric mean $\sqrt[3]{abc}$ that the pupil ought to observe and contemplate. And finally, the geometric mean of two numbers is also the geometric mean of their arithmètic mean and their harmonic mean.

Continuing my search for problems that are close to the pupils' lives and at the same time lead to important algebraic ways of thinking, I considered the parlor games and team games that these pupils play and the dial telephones that link the

lives of so many of them. To find the number of games 5 teams must play in order that each team may play every other team once, to consider the relative advantage of letters and numbers for dial telephones, these and similar inquiries suggest the possibility of introducing pupils early in their second half-year of algebra to the simplest ideas of permutations and combinations. Here is a topic of wide appeal even to laymen. Its few technicalities are not dependent on the rest of algebra. In this topic everyone, old or young, is a beginner.

This then is my suggestion for the second half-year of algebra: that we begin with the progressions, the averages, and permutations and combinations. Seven days, more or less, devoted to each of these three topics would consume four weeks of the second half-year. I would find these four weeks by taking four weeks from the sixteen weeks or more of the second half-year now devoted to factors, fractions, radicals, and the several sorts of equations that I listed earlier.

I recognize the importance of all the material customarily included in this second half-year. I would not injure any part of it. I believe, however, that the progressions, the averages, and the permutations and combinations will come closer to the pupils' lives and will be likely to come back from time to time to these pupils in their adult life in a way that will make them appreciate the brief instruction they once received in these topics.

I believe further that the interest these topics engender will help to retain more pupils in elementary algebra.

Lastly, I believe that the reduction by about one-quarter in the time ordinarily devoted to factors, fractions, and the rest need not operate to rob the pupils of any important value or skill or make these pupils less competent to handle mathematics in college, if I may mention that relatively narrow but by no means trivial objective.

As one who has taught Freshman and Sophomore mathematics at Harvard and Radcliffe for more than 30 years, I am sensitive to the value of a thorough grounding in the fundamentals of algebra. I believe that my colleagues in the Department of Mathematics at Harvard would be content if entering Freshmen had uniformly sure command of the simple fundamentals of algebra. Apart from this, the most glaring weakness that I observe in my pupils with respect to algebra is their relative lack of power to keep on wrestling with a difficult problem. Many of them crumple too soon. They seem not to have been trained to face tasks that demand sustained effort. They give the impression of having been bright enough to do their assignments on the school bus or street car and of having been too rarely confronted with an assignment that could not be disposed of so easily, but really challenged all their powers. This kind of training is not conditioned by choice of topic. It is possible in connection with any topic.

If in my consideration of this second half-year of algebra I seem to some extent to prefer those pupils who threaten to desert algebra as soon as it seems to lose touch with their world, I would ask if any one of the topics I suggest for insertion in advance of factoring, fractions, and the rest is not also of interest and importance for those pupils who are more apt in mathematics and therefore require fewer immediate rewards as stimuli to perseverance in things algebraic.

In beginning this discussion I mentioned a conflict of interests. I had in mind the immediate interests of ninth-grade pupils and also the deferred interest of many of these same pupils that they be adequately prepared for college. My proposal for the second half-year of algebra reflects my sincere desire to resolve this conflict.

And now a few words about the second year of algebra, most commonly taught in the eleventh grade. In some way I should like to see time provided for instruction in the basic ideas of statistics. In making this

suggestion I am guided by the ideas of Dr. A. L. O'Toole in his doctoral dissertation submitted to the faculty of the Harvard Graduate School of Education in 1952. I hope first of all that the instruction in statistics could pay its own way, in part at least, by counting also as instruction in algebra. I am not prepared at this time to go into the academic bookkeeping that must be gone into once this suggestion is taken up seriously. I hope, secondly, that the instruction in statistics would not imitate present courses given at the college level. In the case of high-school pupils the instruction must be more informal and more concrete. I should want it to emphasize a few fundamental ideas only. I should want the pupils to become personally involved in the selection of the inquiries they will make as basis for the statistical procedures they are to learn, and to be personally involved in whatever activities of a statistical sort their inquiries may lead to.

Dr. O'Toole would emphasize from the outset the importance of the ideas "universe" and "sample," and he would lead the pupils to appreciate through their own personal experience the significance of these ideas. For example, from a universe of 5,000 black beads and white beads, he would have the pupils draw many samples of 250 beads each. The drawing of the samples must be carefully supervised to insure that every bead is equally likely to be included in each sample. Indeed, any sampling technique that excludes the operation of the laws of probability is necessarily defective. Beginning with a universe in which the proportion of white to black beads is known, the pupils, by actually comparing each sample with the known universe, acquire the idea of sampling error. Then, using a universe in which the proportion of white beads to black beads is not known, the pupils meet the ordinary practical situation in which a sample is used to afford an estimate of the nature of the universe. In this sort of situation the pupils must recognize the importance of two controlling ideas. The first of these is that the proportions obtained from a large number of samples cluster around the unknown proportion of white to black beads in the universe from which the samples were drawn. The second idea is that the reliability of the proportion that is obtained from a single sample, and that is offered as estimate of the unknown proportion in the universe, can be indicated by some measure of the variation in the clustering (or distribution) of the proportions obtained from a large number of samples. By contemplation of the samples that they have actually drawn from the universe of white beads and black beads, the pupils learn the nature of the risk that is involved in making a conclusion based on a sample from a universe. I omit further details; but you can see the possibility of relating this subject to some elementary ideas concerning probability and to the binomial theorem.

In both my proposal for the second half-year of algebra and in my suggestion of statistics in the second year of algebra, I am trying to conserve all of the traditional algebra that is important to those who will be interested in the physical sciences and at the same time make provision for those who will be interested in the social studies. I believe that this country will have increasing need of persons well trained in each of these broad areas, and I would have the high schools do their best to provide the mathematics essential to each area.

In addition to the foregoing proposals, I wish to add a word or two concerning the recent growth of interest in symbolic logic and abstract algebra without suggesting that we ought to do much about these matters at the high-school level. I cling tenaciously to the opinion that a pupil's first experience with an abstract logical system had better be in connection with demonstrative geometry, where the abstractions can be tempered by reference to pictorial aids. I think it is a rare pupil at the high-school level who can derive bene-

fit from contact with symbolic logic as such. I do wish, however, that the brightest pupils could learn that the logical structure of demonstrative geometry is matched by a logical structure of arithmetic and algebra.

With this in mind, I suggest that some pupils, before quitting their second year of algebra which ordinarily comes after they have studied demonstrative geometry, can consider more closely how negative numbers were introduced into algebra. They will see that the introduction of these numbers was wholly arbitrary, but made with the agreement of all interested parties. I further suggest that these pupils can then think back to the introduction of fractions into arithmetic and can appreciate that these numbers are equally arbitrary in origin and bear the same relation to multiplication and division that negative numbers bear to addition and subtraction. I say nothing about additional time allowance for these ideas. To the extent that they can be communicated, the communication requires not so much labored exposition as deftness by teachers and imagination by pupils.

Let's look at language

Do some of your pupils interchange the words "perimeter" and "area"? It seems that frequently pupils do confound the two, computing an area when the situation requires a perimeter and calculating a perimeter when the problem involves an area.

Among numerous procedures that contribute to effective teaching, the study of the origins of words deserves some attention. What, for example, are the backgrounds of "perimeter" and "area"?

When pupils investigate plane figures, they rightly regard outer boundaries as important. By their outlines, of course, geometric figures are known. A rectangle 10 inches long and 2 inches wide, for instance, differs considerably from a rectangle 5 inches long and 4 inches wide. Applications such as the framing of pictures, the edging of handkerchiefs, and the fencing of fields, abound. Whenever people measure around a figure, they employ ideas for which the Greeks had an appropriate name.

For in Greek "to measure" is *metron* and "around" is *peri*. Accordingly, after pupils have "measured around" a sufficient number of objects they know the idea well enough to accept and retain the name for the idea. Whether the name is Greek or English matters less than pupils' possession of the perimetric principle. Perimeter is around-measure.

Closely similar to the word "perimeter" is the word "periphery." Here the original meanings differ only slightly. Whereas *peri*+*metron* means "to measure around," *peri*+*pherein* means "to carry around."

What did the ancients carry around? Even a modest imagination suffices to adduce some possibilities. Sentries carried their weapons with them around the camp; oxen bore yokes around the field; penitent pilgrims bore burdens around the shrine. Even the surveyors probably had to carry some instruments around to determine an around-measure, or perimeter.

So "periphery" and "perimeter" seem somewhat the same. The latter word, however, remained a technical word, whereas the former acquired several senses, some of them figurative. Although "periphery" can refer to a perimeter, it may also suggest the coastline of Greenland, the surface of an orange, or a vague boundary of human consciousness. In still other contexts it denotes a circumference.

All of which brings us to yet another word. "Circumference" arose as almost an exact Latin counterpart of the originally Greek "periphery." Here the etymology is as simple as *circum*+*ferre*, or "around"+"to bear."

The related Latin noun *circus* designated a ring, oval, or circle. Accordingly, "circumference" at one time in English connoted almost anything round: the outside of a sphere, a spheroidal surface, an arc, even a circuitous course or process!

Figurative uses of "circumference" have not, of course, disappeared entirely. A pupil in a paper averred that "under other circumferences the youth may not have become a criminal." The surprising sequel to this apparent boner is that "circumference" as environment is not at all unreasonable. At least unabridged dictionaries list such a usage.

But by far the most used sense links "circumference" to its close relative, "circle." In fact, the line representing the circumference of a circle is the circle. This meaning, together with the formula $c = \pi d$, is grist for almost any mathematical mill.

Indeed, what would school be like if "area," "circumference," "perimeter," and "periphery" all suddenly disappeared from courses of study?
—*I. H. Brune, Iowa State Teachers College, Cedar Falls, Iowa.*

National enrollments in high school mathematics

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This is a lesson in interpreting statistics of vital interest to those interested in our manpower shortage.

THE CRITICAL SHORTAGE of scientific personnel has caused many persons to seek and quote statistics on the number of pupils enrolled in mathematics. Many times the statements seem to be contradictory. What is the cause of these apparent discrepancies? There may be two basic causes. The data may be out of date, or the data may be misinterpreted.

A glance at the recent data below may cause one to assume that the number of pupils enrolled in algebra has decreased over the years. A second look will inform the observer that the decrease is in percentages.

Per cent of pupils in the last 4 years of public secondary day schools who are enrolled in certain courses in mathematics, 1889–90 to 1954–55.

Year	Algebra	Geometry	Trigonometry
1890[1]	45.4	21.3	1.9
1900	56.3	27.4	1.9
1910	56.9	30.9	1.9
1915	48.8	26.5	1.5
1922	40.2	22.7	1.5
1928	35.2	19.8	1.3
1934	30.4	17.1	1.3
1949	26.8	12.8	2.0
1952–1953[2]	24.6	11.6	1.7
1954–1955[3]	24.8	11.4	2.6

[1] *Biennial Survey of Education in the United States, 1948–50* (Washington: United States Government Printing Office, 1951), Chap. 5, p. 107. (Federal Security Agency, Office of Education)

[2] *Mathematics in Public High Schools*, Bulletin 1953, No. 5, U. S. Department of Health, Education, and Welfare, Washington 25, D. C., p. 34.

[3] Estimate based on this study.

It will be recalled that the number of pupils in high school has increased rapidly over the years; therefore, the percentage of pupils in a subject may have remained constant or even decreased and the number of pupils actually increased. This is true in the case of algebra. The actual number enrolled in algebra (elementary and advanced) was approximately 1,449,000 in 1949, and five years later it was approximately 1,636,000. The per cent of pupils enrolled in trigonometry has remained rather constant, yet the number of pupils enrolled has increased fifteen-fold since 1900.

Also a casual reader of the table might conclude that only about one out of ten high-school pupils takes geometry. Such is not the case. The table indicates that, in the fall of 1954, 11.4 per cent of all the persons in the last four years of high school were enrolled in geometry—plane and solid. Of course, 100 per cent would not be possible, since all pupils were not eligible to take the subject at that time. Certainly most ninth-grade pupils would not take plane geometry, and many eleventh-grade pupils would have taken it the previous year.

Perhaps a less distorted picture of the enrollments in mathematics is given by using the number of pupils in the grade level as a base rather than the total enrollment in the last four years of high school. For example, the number of pupils enrolled in plane geometry in the fall of

1954 was 664,100, which is 10 per cent of the number of pupils enrolled in the last four years of high school; however, this number is 37.4 per cent of the number of pupils enrolled in the tenth grade—the grade in which plane geometry is usually taken. It is true that all pupils enrolled in plane geometry were not from the tenth grade. Many were from the eleventh grade. Likewise, many of the tenth-grade pupils will take plane geometry the following year. However, it is reasonable to assume that at least 37.4 per cent of the pupils who finish high school will have had plane geometry.

Using the grade level enrollment at which the subject is usually offered as the base, the percentage of pupils enrolled in high-school mathematics is as follows:

1. The enrollment in general mathematics (9th grade) is equal to 44.5 per cent of the number of pupils in the ninth grade.
2. The enrollment in elementary algebra is equal to 64.5 per cent of the number of pupils in the ninth grade.
3. The enrollment in plane geometry is equal to 37.4 per cent of the number of pupils in the tenth grade.
4. The enrollment in intermediate algebra is 28.5 per cent of the number of pupils in the eleventh grade.
5. The enrollments in trigonometry and in solid geometry were approximately 13 per cent of the number of pupils in the twelfth grade. (The fact that these subjects are usually half-year courses was considered in this estimate.)

The data above are based on a study[4] of a 10 per cent randomly selected sample of public secondary day schools.

[4] A complete report of this study is available from the Publications Inquiry Unit, Office of Education, U. S. Department of Health, Education, and Welfare, Washington 25, D. C.

". . . eleven, twelve, . . ."

The English cardinal-number words "eleven" and "twelve" appear to be misfits in our number-word system. The number words above ten seem to be formed of combinations, contractions, or words derived from the cardinal words one to ten (with the exception of the powers of ten: hundred, thousand, million, etc.) and eleven and twelve. It should be emphasized that the "teens," when compared to their corresponding Hindu-Arabic numbers, are reversed in order. That is, the units precede the tens in these number words. However, the order of the number words of the twenties, thirties, forties, etc., agrees with that of the corresponding Hindu-Arabic numbers.

But what about eleven and twelve? Why did they become a part of our number-word system? Where did they come from? The answers are found in the Anglo-Saxon language. Anglo-Saxon, sometimes called Old English, is the name of the Germanic language spoken in England between the middle of the fifth and the middle of the twelfth century.

The Anglo-Saxon cardinal-number words *endlefon* ("eleven") and *twelf* ("twelve") are apparently compounded from the cardinals *án* ("one") and *twégén* ("two") or *twá* ("two") and *lifan* ("to leave"). If an individual using a one to one relationship had eleven objects to count, after counting on the fingers up to ten, he would have "one left," *anlif*. Hence, *endlefon* became the Anglo-Saxon cardinal-number word for ten and "one left," or eleven. Similarily, the form "two left," *twálif*, after counting on the fingers up to ten, became the cardinal-number word for ten and "two left," or twelve. The final *on* of *endlefon* ("eleven") may have been added after the analogy of *seofon* ("seven") and *nigon* ("nine").—*Thomas R. Nealeigh, Roosevelt Junior High School, Columbus, Ohio.*

Concerning the function concept

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There are relatively recent demands for a change in the approach to the definition of a function. This paper recalls some of the historical background and presents the arguments for a change.

DURING THE LAST few decades nearly every enterprising author of a mathematics textbook designed for use in the secondary school or in the Freshman year in college has paid conspicuous homage to the function concept. The homage, even though it may have been unduly fulsome and often superficial, had been long overdue. For it has not always been thus. D. E. Smith in his remarkable and encyclopedic *History of Mathematics* (1925) devotes Volume II (725 pages) to "Special Topics of Elementary Mathematics." Although 68 major headings cover elementary mathematics of the traditional sort very thoroughly, the word "function" is not even listed in the index.

There seems no question that in perfecting drill in mathematical techniques, teachers have ignored and perhaps remained unconscious of many broad underlying ideas without which mathematics could lay little claim to its historic inclusion among the humanities as a branch of philosophy. E. H. Moore of The University of Chicago a half-century ago, in pointing out the importance of emphasizing unifying principles such as that of the function concept, succeeded in some measure in reversing the fashion. He would doubtless be disappointed could he note how mild has been the reformation in the general character of mathematics texts. However, other concepts at various logical and semantic levels have enjoyed a recent resurgence of emphasis: mathematics as a mode of communication, approximate measurements, statistical methods, mathematical logic and its special symbolism, set theory, and so forth. But let me confine my remarks to the notion of function.

No one should be surprised to hear that the notion of function has undergone considerable change in recent history. It is, however, naturally highly disturbing to find mathematicians of repute making discordant statements today as to what constitutes a suitable definition of "function." Perhaps a glance backward is in order.

"Function" seems to have been once the equivalent of the word "power" in the sense of algebra. Then the only functions of x were the integral powers x^2, x^3, etc., with the possible inclusion of x^1. Later, more complicated functions, each given by a simple algebraic formula, such as x^2-3x+2, $1/x$, $\sqrt{x^2-1}$, were accepted. Prior to the work of Descartes, little attention was paid to explicit algebraic functions as constituting a family. Of an apparently quite different sort from these algebraic functions even for Descartes, were the trigonometric functions and the then recently discovered logarithms. On the basis of Descartes' emphasis on algebraic form in what had been viewed as pure geometry, it was not many years before Newton gathered together a variety of well-known cubic curves, previously thought independent, each with its own special mechanical method of generation, and supplemented the list by classifying "all" cubic curves. But even then the

equation of a conic section seems hardly to have been thought of as defining a function.

The century and a half covering the work of Euler, Laplace, Legendre, Jacobi, Abel, Gauss, and Riemann (to name a few) saw an impressive and indeed phenomenal rise in the prestige of functions of a complex variable. The earlier discovery that the logarithmic function could be defined as an indefinite integral, Euler's discovery of the identity, $e^{i\theta} = \cos\theta + i\sin\theta$, and of the extension to nonintegers of the factorial function, Gauss' study of complex numbers and of the hypergeometric functions, Jacobi's thorough development of identities and series expansions for elliptic functions, Abel's work on integrals and on solution of algebraic equations of the fifth degree, Riemann's introduction of the visually suggestive Riemann surfaces—all of these helped to suggest that the functions of one or more real variables acquire symmetry and fresh significance when the independent arguments are general complex numbers. Through the use of Taylor series, a common method was at hand for studying functions earlier generated in entirely different ways. The elementary transcendental functions become a recognized class, while elliptic functions, the Gamma function, and the potential functions of Legendre became outstanding examples of higher transcendental functions. Euler's early acceptance of implicit definitions, defined not only by algebraic but also by differential equations, made multiple-valued functions no less acceptable then single-valued functions. Later work by Lie on infinitesimal groups accepted, without qualms, conditions of differentiability. The extraordinary success of Gauss in studying differential geometry seemed added reason for thinking that differentiability is a natural property of all respectable functions. To reject this property, like rejecting the principle of excluded middle, could be expected to lead only to bizarre and fruitless disputation. Euler may be said to have glorified the formula and to have provided by his own success an ideal for many later investigators to follow. The mere existence in the complex domain, of the derivative of a function of a complex variable at a given point, sufficed to assure that the function is analytic. The viewpoint of Euler became part of the cultural background of mathematics even after critics had pointed out its shortcomings. As evidence for this pronouncement, let me cite a 782- (large size) page volume, the second edition of which, published in 1900 by the Cambridge University Press, was entitled (on the cover) *Theory of Functions.* The author, A. R. Forsyth, remarks (p. 14): "It has been assumed that the function considered has a differential coefficient. . . . It has often been called *monogenic,* when it is necessary to assign a specific name; but for the most part we shall omit the name, the property being tacitly assumed. This is in fact done by Riemann, who calls such a dependent complex simply a *function.*"

Not only were the independent variables accepted as being naturally complex variables (save where artificial restrictions to real variables are imposed for reasons of geometrical appeal or physical application), but the total domain of any variable was accepted, and usually tacitly, as the totality of values for which the formula had a meaning. Forsyth in his treatise mentioned above remarks (p. 6) in a footnote: "It is not important for the present purpose to keep in view such mathematical expressions as have intelligible meanings only when the independent variable is confined within limits."

Forsyth considers the view of defining w as a function of z, where w is obtained from z by a sequence of arithmetical operations (and is thus in a literal sense analytic) and hence in such a manner that one can compute the value of w corresponding to any given value of z. Such a definition would have been acceptable to most mathematicians of Euler's time.

Instead, Forsyth announces (p. 8): "A complex quantity w is a function of another complex quantity z, when they change together in such a manner that the value of dw/dz is independent of the value of the differential element dz."

But Euler's tradition had an abundance of sharp critics from early in the nineteenth century. Fourier in his analytical study of heat flow was one of the first to emphasize the arbitrary nature of a function of a real variable. His success and prestige served to emancipate the notion of function from the restriction that an *a priori* representation of it by a single formula is necessary. Fourier seems to have "had almost a contempt for mathematics except as a drudge of the sciences," and felt none of the inhibitions then current concerning functions. He wrote: "We can extend the same results to any functions even to those which are discontinuous and entirely arbitrary." A function no longer needed to be regarded as the embodiment of an algorithm. His boldness taught mathematicians (among his contemporaries was the great Cauchy) that intuitions are often mere prejudices and that even the "obvious" until established by proof may be false. It should have been clear that not every important elementary mathematical function is an analytic function of a complex variable. The complex conjugate, $x-iy$, of the complex variable, $z=x+iy$, is an important nonanalytic function of the complex variable z. Hence also are the absolute value and the real part of z. Euler himself introduced the totient function, $\phi(n)$, of the natural number n, $\phi(n)$ being (for $n>1$) the number of natural numbers each less than n, and relatively prime to n. Euler never made a generalization of $\phi(n)$ to a function of a complex variable.

Of course, any definition is a matter of convention. In a proper sense no definition can be in error inherently except by being self-contradictory. A given formulation may be criticised on many grounds. It may purport to agree with other accepted formulations or to accurately reflect critical usage. In mathematics the most frequent reason for rejecting a traditional phrasing of a definition in favor of new wording seems to be that it has been too restrictive to include situations of newly recognized importance. The old definition then becomes descriptive of merely a part, historically and perhaps continuingly important, but yet not as inclusive as desired for modern research. It would seem entirely appropriate to concentrate attention at secondary-school level to functions expressible by formulas themselves algebraic, or involving at worst the elementary transcendental functions: the trigonometric, inverse trigonometric, exponential and logarithmic functions, and to consider these almost exclusively for real values of the arguments. Other functions may be regarded by secondary-school pupils as "pathological," a term used for them by many mathematicians of the last century.

But let us continue our historical resumé. In revolt against the rather blind and rampant formalism particularly of the German combinatorial school under Hindenburg, Europeans generally began a serious review of the formal methods which had proved so fruitful. Abel and Cauchy were leaders in a critical movement which received great impetus later from Weierstrass and which has almost completely changed the emphasis of mathematical research. Dirichlet in 1837 proposed the following formulation: "A variable y is a single-valued function of the variable x, in the continuous interval (a,b), when a definite value of y corresponds to each value of x such that $a\leq x \leq b$, no matter in what form the correspondence is specified." It may be noted that the favorite phrase "one can find" is missing. The discoverability of the associated value must be inferred, if at all, from the meaning of the term, "definite."

Dirichlet's definition remained standard for a century. It presents difficulties, however. One can ask about the meaning of

the words "variable," "definite," "corresponds." There is the further semantic question as to whether a variable (here y) is itself the function concerned.

W. F. Osgood, in his *Functions of Real Variables* (1936), remarks (p. 68), "At the beginning, we spoke of y as the *function.* Thus we should say: The value of the function, $y=x^2+1$, when $x=1$, is 2. This is a different meaning of the term, but no confusion of ideas arises from these two uses of the word." The question as to whether to accept many-valued functions or to insist instead on several one-valued functions, is partly a matter of taste, but also one of clear thinking. Continued experience had led many mathematicians to confine "function" to the one-valued case. One does not like to see "$\sqrt{4}=\pm 2$." It seems also desirable to reject the ambiguity which Osgood accepted and to define "function" as sharply restricted to the correspondence itself. This involves rejecting the traditional phrase "y is a function of x, etc.," in favor of "y is the value at x, of the function, etc." Such an equation as "$y=\sin x$" is entirely satisfactory, but neither y nor $\sin x$ is the function in this case. The function is the sine function, represented by "sin" and y is its value at x. There is the square function, $(x)^2$, but x^2 is a variable as is x, and writing $y=x^2$ does not make y a function. One could write more formally $y=f(x)$ and define f by "$f:x \rightarrow x^2$," read "f is the function which carries x into x^2." This alone is not enough. One must then explain what values x may take on, as for example, in "$f:x \rightarrow x^2$, (x in R)," provided "R" has been defined as the set of real numbers or rational numbers, as the case may be. The functions "$f:x \rightarrow x^2$, $(0 \leqq x \leqq 1)$" and "$f:x \rightarrow x^2$ (x, an integer)" are not to be confused.

But "correspondence" is itself a vague word, which if used, should be defined. Any thorough-going construction of a mathematical theory must involve from the first the notions of proposition, of set, and of sequence. Is "correspondence" one of those inevitable primitive logical notions hardly explicable in terms at once more simple and more general? Why must functions always have numbers for arguments and numbers for values? In answer to queries such as these, the word "function" is being gradually dropped in some circles, in favor of the less fossilized word "mapping" (borrowed from geometry and group theory), and the case of so-called "many-valued functions," is covered very effectively by emphasizing (and of course defining) the more or less inevitable term "relation."

If A, B are sets, by "$A \times B$" (called the "Cartesian product of A by B") is meant "the set of all ordered pairs (a,b), where a is an element of A, and b of B." By "a two-term relation between variables x,y, having A,B, as respective domains," is meant "a given (non-empty) subset of $A \times B$." At first this definition seems to wreak havoc with literary usage, but one soon becomes accustomed to it and to interpreting all two-term relations in accordance with it. The domain of the two-term relation is the subset D of A, such that for each a in D, there is at least one b in B, for which (a,b) is in the relation. What identifies a function (or mapping) from among other relations is that for each a in the domain D of the function, there is exactly one b in B.

Thus a function of one argument, or a mapping, is simply a one-valued, two-term relation. The term "mapping" thus includes "functional," "projectivity," and so forth. Although the phrase "conformal mapping" is old, the general use here mentioned is very recent and may be due to van der Waerden, 1937.

The evolution of geometry[1]

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The story of the origins of some of our geometrical ideas and the lives of the men who gave birth to these ideas make interesting reading. This is the story of the development of some phases of secondary school geometry.

THE CONSTANTLY CHANGING nature of our concept of geometry has intrigued me for many years. As a student I first visualized geometry, and indeed all of mathematics, as a static body of knowledge which could be mastered if one had sufficient perseverance. As a college student I began to recognize that mathematics was different in the eighteenth century from what it was in the nineteenth or twentieth centuries. When I first became aware of these changes, the elusive nature of mathematics disturbed me. Indeed, I am still a bit disturbed. I shall try to describe to you the high points of the historical evolution of geometry in so far as I understand the geometry of our times.

I hope that the brevity imposed upon me will also keep me from becoming lost in a maze of details. I shall try to present the growth or evolution of our concept of geometry in such a way that we shall not only be aware of the elusive nature of the subject but shall also gain sufficient knowledge of the past to enable us to understand the changes that are in process in our own time. I shall consider primarily the evolution of Euclidean geometry and shall mention other geometries only as they arise through the application of new principles to certain fundamental concepts of Euclidean geometry.

The word "geometry" is derived from the Greek words for "earth measure." This derivation of the word also connotes the origin of the science of geometry. Early geometry was a practical science and an empirical science, that is, a science based upon man's experiences and observations. General theories, postulates, and proofs came much later. Thus our geometry has evolved from a few practical procedures to a deductive science based upon undefined terms, postulates, and the logical deduction of theorems.

We do not know the complete history of geometry. However, we can see several major influences that have contributed to the evolution of geometry. I shall consider very briefly ten of these influences.

1. The empirical procedures of the early Babylonians and Egyptians
2. The Greeks' love of knowledge for its own sake and their use of classical constructions
3. The organization of early geometry by Euclid
4. The embellishment of Euclid's work during the Golden Age of Greece
5. The contributions of Hindu, Arabian, and Persian mathematicians during the Dark Ages in Europe
6. The reawakening in Europe with the growth of the universities, the printing press, and the flowering of all branches of knowledge

[1] Based upon a chapter of the same title in *Fundamental Concepts of Geometry* (Addison-Wesley, 1955) and presented April 15, 1955, at the Thirty-third Annual Meeting of the National Council of Teachers of Mathematics, Boston, Massachusetts.

7. The introduction of coordinate systems and the recognition of the relationship between differentiation and integration, giving birth to calculus in the seventeenth century
8. The application of algebra and calculus to geometry in the eighteenth century
9. The recognition of abstract points, giving rise to many different geometries in the nineteenth century
10. The emphasis upon generalizations, arithmetization, and axiomatic foundations during the past fifty years

As a conclusion I shall endeavor to show how the above influences may be used to predict the influence of our most recent innovation, the machine calculators, and to indicate an appropriate emphasis in our classrooms. Students of history may also observe that the development of geometry is closely related to the development of other areas of mathematics and, indeed, to the development of our culture.

Early measurements

We have evidences of the geometry of the early Babylonians and Egyptians. Records from pictorial tablets indicate that the Babylonians of 4000 B.C. used the product of the length and the width of a rectangular field as a measure of the field, probably for taxation purposes. The pyramids of Egypt provide striking evidence of early engineering accomplishments that probably required the use of many geometric concepts. For example, the granite roof members over the chambers of a pyramid built about 3000 B.C. are 200 feet above the ground level, weigh about 50 tons each, and were probably brought from a quarry over 600 miles away.

The geometry of the early Babylonians and Egyptians was concerned with areas and volumes. Many of our elementary formulas were known in both cultures by 1500 B.C. Undoubtedly some incorrect formulas were used in both cultures. For example, Coolidge[2] states that in both cultures the area of a quadrilateral with sides a, b, a', and b' was expressed as the product of the averages of the pairs of opposite sides $(a+a')(b+b')/4$. We now know that this formula is valid only for rectangles. We must recognize that each of the early formulas was an empirical result to be considered solely on its own merits. There did not exist any underlying body of theoretical geometric knowledge. In general, the Babylonian and Egyptian concepts of geometry appear to have remained at a utilitarian and empirical level until about 600 B.C. when the influence of the Greeks began to have an effect.

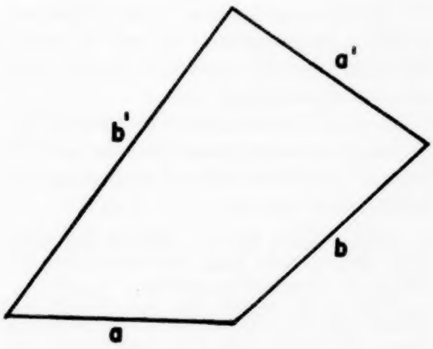


Figure 1

The Greeks

The Greeks with their love of reason and knowledge left a profound impression upon geometry. They encouraged the study of geometry as a science independent of its practical applications. They enlarged the scope of geometry to include not only empirical formulas for areas and volumes but also (1) the use of line segments to represent numbers; (2) the study of properties of polygons and parallel lines; (3) properties of circles and other conic sections; (4) classical constructions with

[2] J. L. Coolidge, *A History of Geometrical Methods*, (Oxford: Cambridge University Press, 1940) p. 5.

straightedge and compasses; (5) ratios and proportions arising from a study of similar polygons; and (6) proofs of consequences of a set of postulates.

Most of the intellectual progress of the Greeks came from schools centered around such outstanding scholars and groups of scholars as

Thales: about 600 B.C., one of the Seven Wise Men of early times and often called the creator of the geometry of lines.

Pythagoras: about 540 B.C., a brilliant scholar and mystic known for a famous theorem which probably provided the basis for the discovery of the irrationality of $\sqrt{2}$ and the corresponding incommensurability of the side and diagonal of a square,

The Sophists: a group of teachers in Athens who were more concerned with theory than practice, who introduced the ideas of the Pythagoreans into Athens, and, among other things, considered the classical construction problems. In particular, they proposed the construction of the trisection of any given angle, the construction of a cube with volume double that of a given cube, and the construction of a square with area equal to that of a given circle.

It is interesting to note here that although these three problems were solved by other methods within a few years, it was over two thousand years before geometry, the theory of equations, the theory of numbers, and the use of algebra in solving geometric problems had developed sufficiently to enable anyone to prove that all three of these constructions were impossible, using only straightedge and compasses.

Finally we consider the school of

Plato about 400 B.C.; who emphasized the recognition of geometry as a part of a liberal education. His school used analysis with reversible steps to discover geometric proofs. His pupil Eudoxus developed the "method of exhaustion" for determining ratios of areas (a forerunner of integration in calculus). Another pupil discovered the parabola, ellipse, and hyperbola as conic sections. A few years later Aristotle's work as a systematizer of logic prepared the way for Euclid's organization of the geometry of his time.

Thus we find the influence of the Greeks in the recognition of geometry as a science independent of its practical applications, the broadened scope of geometry, the recognition of a study of geometry as a part of a liberal education, and the study of geometry as a logical system.

Euclid

The center of mathematical activity shifted from Egypt and Babylonia to the Ionian Islands with Thales, to southern Italy with Pythagoras, and then to Athens. About 300 B.C. we find the center of activity shifting back to Egypt to the newly established university of Alexandria. Euclid was a professor of mathematics at Alexandria. He had probably studied in Athens. He wrote at least ten treatises covering the "mathematics" of his time. His most famous work is called Euclid's *Elements* and contains thirteen books in which he presents an elegant organization of

(1) plane geometry (Books I to IV),

(2) the theory of proportions (Books V and VI),

(3) the theory of numbers (Books VII to IX),

(4) the theory of incommensurables (Book X) and,

(5) solid geometry (Books XI to XIII).

These topics were more closely related than we now consider them. Proportions were based upon similar polygons, the theory of numbers upon the lengths of line segments, and incommensurables upon proportions and the construction of line segments. The books on geometry included nearly all of the concepts that are now considered in a high school geometry course. They also included geometric proofs of algebraic identities and geometric solutions of quadratic equations.

The logical structure of Euclid's proofs was excellent. They included

(1) a statement of the proposition,

(2) a statement of the given data (usually with a diagram),

(3) an indication of the use that is to be made of the data,

(4) a construction of any needed additional lines or figures,

(5) a synthetic proof, and

(6) a conclusion stating what has been done.

We cannot be sure whether this logical structure is due primarily to Euclid or to his training. In either case it is a consequence of the trend in early Greek philosophy.

Lest we idealize Euclid, we should recognize that he adopted many Greek ideas including Aristotle's distinction between postulates and common notions; that he often used definitions as descriptions of terms; and that he tacitly assumed the existence of points and lines, order relations on a line, continuity, and the infinite extent of a line. However, the results of two thousand years of experience should not seriously detract from the significance of Euclid's *Elements*.

Euclid's first three postulates are probably Plato's assumptions for classical constructions. However, Euclid's geometry included much more than classical constructions. He simply used the constructions to demonstrate the existence of the points and lines under consideration. The fourth postulate concerned the equality of right angles; the fifth postulate was the parallel postulate. Apparently Euclid felt uneasy about the fifth postulate since he avoided using it as long as possible. We shall find that this skepticism was justified and was shared by other mathematicians. In general, Euclid's *Elements* rendered geometry a tremendous service as an organization of geometry and indeed of all the mathematical knowlledge of that time. We do not know how much of this material was original with Euclid. We do know that the *Elements* represents a logical outgrowth of the geometry of the early Greeks.

Early Euclidean geometry

The word "mathematics" is derived from the Greek word meaning "subject of instruction," and at the time of the Pythagoreans referred to geometry, arithmetic, music, and astronomy. As knowledge increased, the concepts of mathematics and geometry were restricted. For example, optics, surveying, music, astronomy, and mechanics gradually became recognized as separate bodies of knowledge. Algebra, trigonometry, and the theory of numbers became separate branches of mathematics. Today such specialization is decreasing since it is not unusual for a modern research problem to arise in classical algebra, have significance in geometry, and be solved using the theories of analysis. From this point of view, our concern with geometric concepts (in a narrow sense) is against the modern trend.

Since Euclid's *Elements* contained a logical organization of nearly all the mathematical concepts of his time, it provides a basis for a consideration of the mathematical achievements of the next thousand years. The first half of this period is characterized by a continued development of mathematical concepts, the second half has decreasing mathematical significance. At the end of this period Alexandria was destroyed by the Arabs, and Europe was entering its Dark Ages.

The prestige of Alexandria lasted for many years after Euclid. Archimedes (about 250 B.C.) studied there and, in addition to discoveries in other fields, made many contributions to geometry. For example, his work on areas and volumes of revolution provided another forerunner of integration. Apollonius (about 225 B.C.) studied at Alexandria and wrote an extensive treatise on conics. His descriptions of figures in terms of a diameter and a tangent line were equivalent to a coordinate system. He recognized that the distances of a point from the two foci have a constant sum for the points on an ellipse, a constant difference for the points on a hyperbola. Later, about 300 A.D., Pappus enhanced the prestige of Alexandria through his work on volumes of revolution, the construction of conics through given sets of points, and the study of special curves including spirals.

The remaining mathematicians of note prior to the destruction of Alexandria were

primarily commentators upon the works of their predecessors. Thus by 700 A.D. the geometry of measurements of the Babylonians and Egyptians had been molded in accordance with the Greek love of knowledge and reason and was in need of new influences. This is not intended to minimize the Greek influence but rather to indicate that it had run its course.

Hindu, Arabian and Persian influences

Each culture that passes along a body of knowledge makes a contribution to that knowledge. Thus, especially during the Dark Ages in Europe, the mathematical achievements of the Greeks were modified by Hindu, Arabian, and Persian mathematicians. Most of these influences were of a practical and utilitarian nature There were noteworthy advances in number notation, areas, volumes, classical constructions, astronomy, and trigonometry. Euclid's parallel postulate was seriously questioned. Omar Khayyám, the author of the *Rubáiyát*, wrote a treatise on algebra and determined roots of some cubic equations as intersections of conics. Like their Greek predecessors, these mathematicians used line segments to represent numbers and therefore did not recognize "negative" roots. Also like their predecessors, they proposed replacements for Euclid's parallel postulate, but failed to prove either the postulate or their replacement as a theorem. We now know that the parallel postulate cannot be proved as a theorem unless an equivalent statement is taken as a postulate.

Throughout the development of Euclidean geometry we shall find that the attempts to prove the fifth postulate were generally of one of the following types: (1) direct proof from Euclid's other postulates; (2) replacement of the fifth postulate by a more "self-evident" postulate (either explicitly or tacitly and unknowingly) and a proof of the fifth postulate as a consequence of the new assumption and Euclid's other postulates; or (3) indirect proof by showing that the fifth postulate cannot fail to hold. All of these approaches were doomed to failure as "proofs" of the fifth postulate. They did, however, encourage the study of several possible geometries using primarily (1) synthetic methods; (2) algebraic concepts, (3) curvature and differential geometry; (4) distance relationships; and (5) groups of transformations. As a result of these studies, mathematicians have considered the axiomatic basis for Euclidean geometry in detail, and now recognize the consistency of the non-Euclidean geometries.

The reawakening in Europe

In mathematics as in trade and art, the first signs of European awakening were in Italy. There were some evidences of mathematical life in England in the eighth century and in France in the tenth century. However, progress was very slow. About the end of the twelfth century Leonardo returned to Italy after traveling extensively, and soon published several treatises making available a vast amount of information regarding previous achievements in number notation, arithmetic, algebra, geometry, and trigonometry. These ideas were soon picked up in the new European universities where groups of scholars provided the stimulus to each other that is necessary for great achievements. Then in the fifteenth century the printing press provided a new means of disseminating knowledge, and intellectual activity began to spread rapidly throughout Europe. At first there was an intellectual smoldering while the scholars acquired additional knowledge of past achievements. Soon Alberti and the Italian artists developed some of the principles of descriptive geometry. A treatise on plane and spherical trigonometry was prepared in Germany. Letters were introduced for numbers in France. Gradually mathematical activity acquired momentum and by the end of the seventeenth century was well under way. Dur-

ing the last three centuries there has burst forth such an avalanche of activity, constantly expanding without any definite signs of spending its force, that we shall be hard pressed to assess the evolution of geometric concepts. However, I shall make the attempt and let you be the judge of the success of the venture.

The seventeenth century

Kepler and Galileo made their contributions to geometry and astronomy at the beginning of the seventeenth century. One of Galileo's pupils, Cavalieri, assumed that a line could be generated by a moving point, a plane by a moving line, a solid by a moving area. Many of you have taught *Cavalieri's principle* regarding the equality of the volumes of any two solids of the same height and of the same cross-sectional areas at corresponding heights.

Our Cartesian coordinate systems are based upon the work of René Descartes who applied algebraic notation to the analysis of conics by Apollonius, visualized all algebraic expressions as numbers instead of geometric objects, and found equations representing several curves (considered as loci). His interpretation of such symbols as x^2 and x^3 as numbers and, therefore, as lengths of line segments was very important. Previously, linear terms such as x or $2y$ had been considered as line segments, quadratic terms such as x^2 or xy had been considered as areas, and cubic terms such as x^3 or x^2y had been considered as volume. The old interpretations were restrictive in the sense that only like quantities could be added. For example, it was permissible to add x^2 and xy (areas), but it was not permissible to add x^2 and x (i.e., an area and a line segment). Descartes' interpretation of all algebraic expressions as numbers and therefore, as line segments made it possible to consider sums such as x^2+x. This new point of view provided a basis for the representation of curves by equations.

The study of geometric figures as loci corresponding to equations introduced a new era and an entirely new point of view into the study of geometry. Since we shall find enthusiastic supporters of both the new and the old points of view, we shall endeavor to distinguish between the two as follows: The study of figures in terms of their algebraic representation by equations will be called *analytic geometry*; the study of figures directly without using their algebraic representations will be called *synthetic geometry*. Since many geometers will use both algebraic and synthetic methods, the above distinction is at best a relative one. Modern geometers consider the two geometries as two equivalent points of view of the same body of knowledge.

Fermat was a contemporary of Descartes who also worked in analytic geometry. Desargues and Pascal visualized the circle, ellipse, parabola, and hyperbola as projections of circles, discovered other properties of conics, and prepared the foundations for synthetic projective geometry.

The last half of the seventeenth century is marked by the independent discovery by Newton and Leibniz of the relationship between differentiation (often visualized as rate of change) and integration (often visualized as summation). Newton's discovery of calculus was based upon a study of figures in motion, Leibniz' upon a study of static figures. Our present notation is largely that of Leibniz. Newton also made contributions to analytic geometry through his classification of cubic curves and to synthetic geometry.

The application of new ideas to geometry typifies the seventeenth century. There was the application of algebra initiated by Descartes and Fermat, the applications of projections initiated by Desargues and Pascal; and the applications of calculus initiated by Newton and Leibniz. These new approaches were the beginnings of three major phases of geometry—analytic geometry, synthetic geometry, and differential geometry.

The eighteenth century

In the eighteenth century we find a broadening of the applications of the new ideas started in the seventeenth century and a new development in relation to Euclid's parallel postulate.

The work of Descartes and Fermat was extended to three dimensions. Previously known results and new results were restated in algebraic notations. Transformations of coordinates were considered.

Newton's work was continued by Maclaurin who discovered many special curves, including the cissoid, cardioid, and lemniscate. The techniques of differential geometry were used to develop a theory of curvature and to study quadric surfaces.

In synthetic geometry Euler proved that for any given triangle the point of intersection of the altitudes, the point of intersection of the medians, and the point of intersection of the perpendicular bisectors of the sides are collinear. He also considered networks of arcs and developed a general theory of traversability which is often considered as the starting point of topology.

The work on Euclid's parallel postulate was another development of synthetic geometry. Early in the eighteenth century Saccheri tried to prove the postulate by denying it and showing that the results so obtained could not hold. He succeeded in proving that exactly one of three situations must consistently hold:

1. the sum of the angles of a triangle is always equal to two right angles,
2. the sum of the angles of a triangle is always greater than two right angles, or
3. the sum of the angles of a triangle is always less than two right angles.

Under each hypothesis he could prove several theorems. He disposed of the last two hypotheses by accepting Euclid's tacit assumption that lines are not reentrant and by assuming that two lines cannot merge into a single line at infinity. We now recognize these assumptions as postulates leading to independent geometries. We know that the first hypothesis holds for plane triangles, the second for spherical triangles. Lambert (still in the eighteenth century) suggested that the third hypothesis might hold for triangles on a sphere with an imaginary number as its radius. This suggestion is now recognized as mathematically sound. Lambert based it upon his study of the geometry on the surface of a sphere. As a part of this study he proved that the area of a spherical triangle was a function of the excess of its angle sum relative to two right angles.

Saccheri's approach was also adopted by Legendre (1752–1833). However, he too was trying to prove the parallel postulate. Legendre is well known for his work in the theory of numbers, theory of functions, and calculus. He has had a great influence upon American high school geometry texts through his publication of a geometry text rearranging and modifying *Euclid's Elements*.

During the last half of the eighteenth century we find increasing evidence of the forthcoming crescendo of mathematical activity that characterizes the nineteenth and twentieth centuries. Because of the increased specialization of terminology and the detailed and abstract character of many of the contributions, our treatment of the evolution of geometry will, of necessity, become more expository.

The nineteenth century

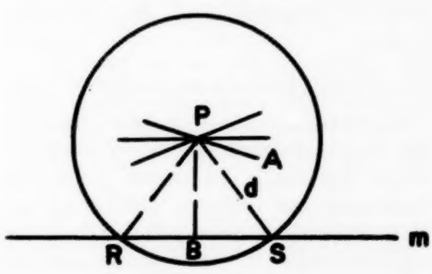
In the early nineteenth century Gauss apparently visualized the possibility of geometries in which Euclid's fifth or parallel postulate did not hold. However, he did not publish his results. Probably he observed that Euclid's fifth postulate is quite different from his first four postulates. The first four postulates are concerned with finite segments of lines and the possibility of extending a finite segment to form a line. The fifth postulate

asserts a property of lines in their full extent. Thus, since all measurements and constructions are finite (even the most distant star visible in the most powerful telescope is only a finite distance from the telescope), the fifth postulate is based upon a faith or conviction that if two lines are cut by a transversal such that the sum of the interior angles on one side of the transversal is less than two right angles, then the lines will intersect if they are extended sufficiently far. When the sum of the interior angles differs very slightly from two right angles, the assumption that the lines will intersect cannot be definitely established.

Given a line m and a point P that is not on m, we may draw a circle with center P and any given radius d intersecting line m in points R and S. Then it is clear that there are infinitely many lines through P that do not intersect m inside the circle, i.e., within the distance d from P. When d is the radius of the circle that I have drawn, these non-intersecting lines are expected. Why shouldn't there also be infinitely many lines through P and not intersecting m when d is a mile, the distance to the horizon, the distance to the moon, the distance to the most distant visible star? This question could not be answered 150 years ago. It is now known that one may obtain consistent geometries by assuming either

1. there are infinitely many lines through P that do not intersect m, or
2. there is exactly one line through P that does not intersect m.

Figure 2



If the line PB is perpendicular to m, then Euclid's fifth postulate asserts that any line PA such that angle BPA is less than a right angle must intersect the line m if the lines PA and m are extended sufficiently far.

Bolyai and Lobachevski independently developed theorems in a geometry based upon the assumption that there are infinitely many lines through P that do not intersect m. Later this geometry was called *hyperbolic geometry* in recognition of the two distinct lines, PR and PS, "parallel" to the line m. Hyperbolic geometry corresponds to Saccheri's third hypothesis that the angle sum of a triangle is less than two right angles.

Riemann developed a geometry based upon Saccheri's hypothesis that the angle sum of a triangle is greater than two right angles. He considered space as a set of undefined objects, called "points," where each point was determined by its coordinates. He assumed that there existed a distance function and that the square of the differential of the distance function was homogeneous and of the second degree in the differentials of the coordinates. For example, in Euclidean plane geometry the distance function

$$s^2=(x_1-x_2)^2+(y_1-y_2)^2$$

may be expressed in the form

$$ds^2=dx^2+dy^2$$

in terms of the differentials of the distance and the coordinates. Then just as we consider transformations (rotations and translations) in Euclidean geometry that do not change the Euclidean distance function, Riemann considered transformations that do not change his generalized distance function. Thus, as in Euclidean geometry, Riemann assumed that the measurements of figures and objects do not depend upon their position. Accordingly, on any surface the figures must be freely movable as on a plane (on which figures may slide or rotate) or on the surface of a sphere.

The distance functions or metrics were developed further by Cayley and Klein. The new theories were extended to three dimensions and then to n-dimensions. Klein also classified various geometries by considering them as studies of properties that are invariant (unchanged) under groups of transformations. Lie considered groups of transformations leaving distance functions invariant and proved that if figures are to be freely movable (slide in any direction and rotate about any point), there are exactly four possible types of geometry in three-dimensional space. These are: (1) Euclidean geometry; (2) the geometry on a sphere; (3) the geometry of Riemann; and (4) the geometry of Bolyai and Lobachevski. The last two are often called the *non-Euclidean geometries.*

The use of algebra (especially the theory of groups) and differentials in the development of the above geometries provides another illustration of the interdependence of the various branches of mathematics. This interdependence is also seen in the search for proofs of the consistency of geometries. It can now be proved that Euclidean geometry is consistent if the real number system is consistent. (This dependence of geometry upon properties of numbers is an example of the arithmetization of modern geometry.) The non-Euclidean geometries are also consistent if the real number system is consistent. Accordingly, even though several prominent philosophers have based their beliefs in natural logical systems upon the existence of a natural geometry—Euclidean geometry—we must now recognize that Euclidean geometry may not be the inherent geometry of our universe. It is probable that the development of philosophy as well as mathematics would have been noticeably changed if this had been recognized several centuries ago.

Let us now return to the early nineteenth century. At the end of the eighteenth century Monge was active in France. He organized descriptive geometry and applied the new techniques of algebra and calculus to curves and surfaces. One of his pupils, Poncelet, wrote the first text on projective geometry and developed the concept of duality. Von Staudt considered a geometry of positions independent of all measurements and proved that geometry did not need the techniques of algebra and calculus. Gauss' ideas on curvature were extended by Riemann and eventually provided a basis for relativity theory. Riemann also used n sheets (in a sense, planes) to render n-valued functions single-valued on these new Riemann surfaces. This concept had important implications in both the theory of functions and geometry. Plücker, Cayley, and Grassman developed new coordinate systems and extended their results to n-dimensional geometries. Moebius introduced new coordinates and discovered a new type of surface, a one-sided surface.

As indicated by the introduction of several types of coordinates, by Riemann's concept of a point as an undefined entity determined by its coordinates, and by the introduction of n-dimensional spaces, the concept of a point was undergoing a change in the middle of the nineteenth century. The complete abandonment of all visual intuitive concepts of a point was gradually accepted by theoretical mathematicians. The new concepts of a point were used in the non-Euclidean geometries and as a basis for abstract geometry. Thus near the end of the nineteenth century we find geometry extending its break with the physical world. This break was completed with the axiomatic developments of the twentieth century.

The twentieth century

About the end of the nineteenth century the Italian geometers enjoyed a period of active leadership. In recent years this leadership has probably shifted to the United States with the influx of scientific immigrants and the development of topol-

ogy and abstract algebraic geometry. Topology is a very general and important geometry. In many ways it represents the peak of our geometric achievements to date. It also involves several elementary concepts that could be useful in secondary schools. Some of these concepts are described in the November, 1953 issue of THE MATHEMATICS TEACHER.[3]

The shift in leadership in algebraic geometry appears to be due to the introduction of the arithmetic ideas of Dedekind and Weber and the modern algebraic concepts of group, ring, field, and ideal. Since the trend in algebraic geometry is indicative of trends in other branches of geometry, I shall present the following introductory remarks of Oscar Zariski in a paper presented at the 1950 International Congress of Mathematicians in order to give the viewpoint of one of the leaders in this area.

> The past 25 years have witnessed a remarkable change in the field of algebraic geometry, a change due to the impact of the ideas and methods of modern algebra. What has happened is that this old and venerable sector of pure geometry underwent (and is still undergoing) a process of arithmetization. This new trend has caused consternation in some quarters. It was criticized either as a desertion of geometry or as a subordination of discovery to rigor. I submit that this criticism is unjustified and arises from some misunderstanding of the object of modern algebraic geometry. This object is not to banish geometry or geometric intuition, but to equip the geometer with the sharpest possible tools and effective controls. It is true that the lack of rigor in algebraic geometry has created a state of affairs that could not be tolerated indefinitely. Effective controls over the free flight of geometric imagination were badly needed, and a complete overhauling and arithmetization of the foundations of algebraic geometry was the only possible solution. This preliminary foundational task of modern algebraic geometry can now be regarded as accomplished in all its essentials.
>
> But there was, and still is, something else more important to be accomplished. It is a fact that the synthetic geometric methods of classical algebraic geometry, operating from a narrow and meager algebraic basis and faced by the extreme complexity of the problems of the theory of higher varieties, were gradually losing their power and in the end became victims to the law of diminishing returns, as witnessed by the relative standstill to which algebraic geometry came in the beginning of this century. I am speaking now not of the foundations but of the superstructure which rests on these foundations. It is here that there was a distinct need of sharper and more powerful tools. Modern algebra, with its precise formalism and abstract concepts, provided these tools.
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> An arithmetic approach to the geometric theories which we were fortunate to inherit from the Italian school could not be undertaken without a simultaneous process of generalization; for an arithmetic theory of algebraic varieties cannot but be a theory over arbitrary ground fields, and not merely over the field of complex numbers. For this reason, the modern developments in algebraic geometry are characterized by great generality. They mark the transition from classical algebraic geometry, rooted in the complex domain, to what we may now properly designate as *abstract algebraic geometry*, where the emphasis is on abstract ground fields.[4]

We have seen in our previous discussion and in the above quotation that the trend of geometry in the first half of the twentieth century has been toward generalization, arithmetization (i.e., the use of properties of numbers), and the axiomatic foundations of geometry. Pasch visualized geometry as a deductive science based upon a set of postulates. Hilbert and Veblen were especially active in this formalized concept of geometry. Most of us slyly appreciate Hilbert's concept of mathematics as a game played according to certain rules with meaningless marks on paper.

CONCLUSION

This paper has been concerned with the origins of geometry. What about the future? Are we steadily pushing back the horizons of mathematical knowledge? Are we gaining an ever firmer grasp upon this body of knowledge, its underlying principles, and its implications? Such appears to be the case. Moreover, let us not lose sight of the manner in which this progress is being made—new mathematical tools

[3] B. E. Meserve, "Topology for Secondary Schools," THE MATHEMATICS TEACHER, Vol. XLVI (November 1953), pp. 465–474.

[4] Oscar Zariski, "The Fundamental Ideas of Abstract Algebraic Geometry" *Proceedings of the International Congress of Mathematicians*, Vol. II, 1950, p. 77.

are being combined with an emphasis upon generalizations and fundamental operations. The secondary schools cannot hope to introduce all the new tools; the schools could and, in my opinion should, place a major emphasis upon the importance and use of generalizations and the fundamental operations.

In geometry the influence of intellectual curiosity has been evident since the time of the early Greeks. Many phases of geometry have been primarily intellectual achievements and have not found a practical application until they were well developed. In other cases utility came earlier in the development. Recently, technological advances have provided a wide range of activity for the mathematically curious as well as those who are trying to accomplish specific tasks. The electronic and mechanical computers provide a striking example of the influence of technological advances. Within the past decade these new mathematical tools have gained recognition for the ways in which they may remove most of the drudgery from mathematical computations in both applied and theoretical branches of mathematics. Within another decade it seems reasonable to predict that these machines will revolutionize many of our approaches to mathematical problems. This new technique appears destined to go through the same stages that we have observed in the use of the techniques of algebra and calculus. At present we are in the period of the discovery of the technique and the initial explorations. Problems of rigor are arising. We may expect a rash of exploitations of the new technique comparable to the exploitations of calculus in the eighteenth century. As in the case of algebraic and synthetic methods in geometry, there will undoubtedly be strong proponents of old and new techniques. Also as in the case of algebraic and synthetic methods in geometry, we may look forward to the eventual absorption of the new technique into accepted mathematical procedures with a resulting unification of viewpoints and increase in our knowledge of mathematics.

Scientific advances in many phases of our society are creating a very fertile source of problems for calculating machines. The greatest problem at present is the preparation of the problems for the machines. As the president of a nationally known research corporation said recently: "We have not run out of problems, we have run out of mathematicians." Here again the secondary schools can serve their students most effectively by emphasizing fundamental operations and basic procedures. Throughout the evolution of geometry and, indeed, all of mathematics we find progress dependent upon people who understand basic operations and have the courage to try new ideas. Geometry today represents but a fleeting instant of an ever-changing body of knowledge. Even in our minute personal worlds, we may have an influence upon the evolution of geometry by keeping ourselves aware of the changes that are taking place and by transmitting this awareness to our students through our teaching of mathematics as a living subject, based upon generalizations of the fundamental operations and procedures that we develop in our classrooms.

Thus you see that I do not consider our hasty journey through the history of geometry as a momentary intellectual pastime. Rather I find in such journeys both inspiration and guidance for my daily classroom teaching. I find a living subject, a subject of great importance to our national and cultural welfare, a subject whose growth depends upon the efforts of thousands of people like you and me who must keep trying to teach the underlying principles behind the formal details in our textbooks and to keep our students active in the formulation and testing of their own ideas and generalizations.

Mathematics and the changing curriculum of post-war Japan[1]

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The problems involved in reconstructing the schools of Japan were difficult ones. Here is a first-hand account of how some of the problems were solved and how others were left unsolved.

IN MAY, 1951, a request came from the U. S. Department of the Army, asking that I accept their invitation to serve as Mathematics Consultant to the Japanese universities in connection with the Institute for Educational Leadership under the control of the Civil Information and Education section of the Supreme Commander for the Allied Powers.

At first I could find many reasons for not wanting to go to Japan. They were the usual reasons—the danger involved, family separation, professional commitments, and so forth. But, in spite of myself, the call for help from a country eager to learn the ways of democracy and eager to adapt their educational system so as to make democratization possible became a professional challenge too hard for me to resist. Two months later I accepted the call. Today I am happy I did.

It would be hopeless for me to cover the entire scope of my work in Japan in the course of one lecture. Hence, I shall confine my comments to the following areas:

1. The seventh IFEL Mission to Japan:
   a. its meaning, purpose, and plan of operation
   b. my assignment in this mission
2. A comparison of Japan's school system before and after the war
3. The pre-war curricula and post-war changes
4. How the IFEL groups (in mathematics) operated
5. Some accomplishments; and some unsolved professional problems

I. IFEL, ITS MEANING, PURPOSES AND MODE OF OPERATION

The abbreviation stands for "Institute for Educational Leadership." General MacArthur's policy with regard to Japan's reconstruction (including its educational system) was, in essence, "Do nothing for the Japanese which they can do for themselves." This did not mean total abandonment nor the creation of a hopeless feeling of despair. On the contrary, it was a noble attempt to boost the morale and confidence of the Japanese in their own abilities and responsibilities for the rehabilitation of their country.

This policy proved to be a wise one. Japanese leaders arose, after the necessary screening and purging, who worked enthusiastically with the CIE (Civilian Information and Education) section of SCAP (Supreme Commander for Allied Powers).

Following the first U.S. Education Mission to Japan in 1946, and in keeping with its recommendations for reform of the Japanese educational system, a series of IFEL's were held in selected cities throughout Japan.

[1] Presented at the Mathematics Institute, Louisiana State University, Baton Rouge, Louisiana, June, 1955.

The first IFEL's were designed to prepare Japanese leaders for administrative and supervisory posts: superintendents of schools, supervisors, professors of education, youth activity leaders, and so forth.

It must be remembered that all prewar educational directives came from the Ministry of Education (*Mombusho*) in Tokyo. Directly after the war more and more leadership was to come from the local administrators—superintendents, principals, professors—and less and less from the central headquarters of education in Tokyo. To develop this badly needed leadership was no easy task, as you can well imagine. However, the accomplishments made under the most trying circumstances were really remarkable.

The Seventh and Eighth IFEL Missions (1951–52) were the last ones to be conducted by CIE of SCAP. These last two missions were sponsored jointly by the Japanese and our CIE section of SCAP. They were organized at the request of the Japanese and designed to develop leadership in specialized subject-matter fields such as science, mathematics, social studies, industrial arts, nursing, education, curriculum design, and so on.

The Japanese Ministry of Education and university officials asked SCAP for an American Mathematics-Education Consultant to help them reorganize their mathematics program, including the elementary schools, the junior and senior high schools, and the teachers' colleges. The call came from the U.S. Department of the Army in May, 1951. I finally accepted the appointment in July and was on the job in Tokyo by the last week in August.

The two mathematics groups of the Seventh IFEL consisted of thirty to thirty-five members each. The first group met at Hiroshima University in a six-weeks intensive workshop; the second mathematics group met at the Tokyo University of Education for an equal length of time. All of these men (and one lone woman) in the IFEL groups were selected to represent their prefecture, university, or teachers college after a careful screening of their applications. Certainly they were among the best professors of mathematics-education, or aspirants to such positions, in teachers colleges and universities of Japan. Their purpose was to find solutions to the problems arising out of the rapid democratization of the schools and colleges and the reorientation of mathematics in this new and rapidly changing program. It therefore became part of my duty to help them in arriving at solutions to their immediate problems and to plan dynamic programs in elementary, and secondary school mathematics—as well as a teacher-training program to make possible the achievement of these aims. This was a gigantic task, the most stimulating and professionally rewarding assignment I have ever attempted.

II. Japan's school system before and after the war

To appreciate the problems of curriculum reorganization in Japan, we should take a quick look at the organization of the schools prior to 1945, and the abrupt changes in the organization since the close of the war.

Japan's school system, up to 1945, consisted of a six-year elementary school (*Sho-Gakko*), open to all children.

1. At the end of the sixth grade, pupils had to choose between attending a higher elementary school for the seventh and eighth grades (*Koto Shogakko*), or going directly into some kind of labor or industry market.

2. If they chose this type of school, they had an additional choice of attending a part-time continuation school (*Seinen Gakko*) for the equivalent of the ninth and tenth grades.

3. If children wanted to, they could elect to attend a *Jitsugyo* (vocational school) after the seventh and eighth grades in the higher elementary school,

or they could go directly from the elementary school (*Sho-gakko*) at the end of the sixth grade into the vocational school for the next five grades (7–11).

4. From these vocational schools pupils could find semi-skilled jobs or continue their education in grades 12–14 in a *Semmon Gakko* (college) or *Shihan Gakko* (normal school), leading to semi-professional positions (semi-technical work or teaching in elementary school).

5. At the end of the sixth grade, Japanese girls for the most part entered the *Koto Jogakko* for an additional five years, after which they usually married into upper- or middle-class homes.

6. If a young Japanese who had just finished the sixth grade decided to strive for a clerical position or semi-professional position, he would continue his education for an additional five years in *Chu-Gakko* (middle school).

7. If he decided at the end of the eleventh grade (after graduation from a *Chu-Gakko* or middle school) to enter a university (*Daigaku*), he had to choose between attending a *Daigaku Yoka* (university preparatory course) or a *Koto Gakko* (higher school). Each of these courses extending through the twelfth, thirteenth and fourteenth grades.

8. After the successful completion (by passing university entrance examinations) of the 14th year in a university preparatory or higher school, students continued their study in a three-year university course. This led directly to high professional positions.

9. A small number, after university graduation, continued in the university graduate school from one to three years.

This, in brief, constitutes the pre-war Japanese school system. If it is possible to visualize such a complex system in relation to problems of finance, school housing, teaching personnel, educational equipment, guidance services, and the like, perhaps one can also understand why the curricula in these various schools were so inflexibly designed.

III. The pre-war curricula and post-war changes

The curriculum of the Japanese schools was rather poorly constructed. It consisted mainly of subject-matter courses, many in number and some being taught only once or twice each week. Courses that were thought worthwhile for some were made mandatory for all. Hence, the number of required courses prevented the notion of an elective system being considered seriously in pre-war Japan.

Following the Potsdam Proclamation and the surrender terms, the objectives of education changed radically and the curriculum necessary for achieving the new goals had to be developed. This was an enormous undertaking. The old pre-war methods of curriculum construction and the development of courses of study were no longer suitable to produce the desired results. The new guide for curriculum building was the following statement issued by the U.S. Education Mission in 1946:[2]

> A good curriculum cannot be designed merely to impart a body of knowledge for its own sake. It must start with the interests of the pupils, enlarging and enriching those interests through content whose meaning is intelligible to the pupils. As in the statement of aims, so in the construction of the curricula and courses of study, the pupil in a particular environment must be the starting point. This principle is violated if a central authority issues an educational currency warranted valid under all circumstances irrespective of the environment and activities of the pupils.

It was quite obvious that no hurried job of curriculum patching would suffice. Hence, during the school years of 1945–46 and 1946–47 the old form of subject-matter curriculum continued in use. Meanwhile, the Ministry of Education (*Mombusho*), through its Curriculum Committee and representatives from the Bureau of School Education and Bureau of Textbooks, worked very closely with SCAP advisers in laying the foundations for curriculum construction: (1) the general aims of education, (2) the specific aims

[2] "*Education in the New Japan,*" Vol. I, Text (Tokyo: May 1948). P. 186.

of each subject area, (3) various stages in the psychological growth and development of children, (4) various interests and social needs of children, (5) social activities among children, and (6) community-life activities. The 1947–48 tentative courses of study included many ideas suggested by the U.S. Education Mission. Though these ideas were developed speedily, they were quite respectable pieces of professional work. They were predicated upon two resolutions:

1. Special textbooks and special courses for girls only would cease to exist.
2. In all types of schools, children of the same grade level would follow the same basic courses of study in areas such as language, social studies, etc.

These reforms eliminated, as far as the curriculum was concerned, the discriminating practices preventing pupils with certain "types" of preparation from entering secondary schools of their choice. These reforms should be marked as important steps in the democratization of Japan's post-war schools.

IV. How the IFEL groups operated

As mentioned before, the first six IFEL Missions to Japan sponsored by CIE-SCAP have done much to aid the leaders in Japanese education to plan the program for a democratic school system. It must be remembered that the Japanese themselves have co-operated extremely well in the planning and the execution of the plans jointly agreed upon. In a real sense the proposals for reform represented the fruits of Japanese thinking and planning under guidance of CIE-SCAP.

The next step in the development of educational leadership among the Japanese was to consider the problems of curriculum development as they relate to special subject areas such as mathematics, science, social studies, industrial arts, nursing education, and so on. The Seventh and Eighth IFEL's were designed for this purpose. The Seventh IFEL was sponsored by the Japanese Ministry of Education and the universities themselves with a minimum of supervision and control by CIE-SCAP.

Hence, when the Seventh IFEL mission arrived in Japan, the Japanese IFEL leaders in mathematics presented me with tentative plans for intensive six-week workshop sessions to be held at Hiroshima and Tokyo. The preliminary plans were considered tentative—subject to review, discussion, and approval by the group participants and the American consultant.

At the first meeting it became clear that a series of lectures (as the tentative plans called for), given either by the American consultant or the best mathematics-education leaders in Japan, was not the most suitable approach to use in arriving at solutions to their problems—especially *if leadership among the participants was to be developed.* Instead it was decided that the six-week study sessions were to be conducted in *workshop* fashion.

1. Each member of the group submitted a list of professional problems to which he was seeking solutions.
2. These problems were then classified and put into the following categories:
   a. Aims and purposes for teaching elementary and secondary school mathematics.
   b. Selection and grade-placement of subject matter.
   c. Methods of teaching.
   d. Evaluation.
   e. Teacher preparation in mathematics and education.
3. Each participant elected to work in one or two of these areas.
4. a. Morning sessions were devoted to group discussions and planning, or to lectures by specialists on certain crucial problems. (The lecturers who were previously engaged now served as experts and

resource persons upon request of the planning committee.)

b. The afternoon sessions were devoted to small-group study and research.

c. During those sessions a regular schedule of semi-private conferences was arranged for the purpose of developing closer professional contacts between the American consultant and members of the workshop groups. These proved to be exceptionally worth while in meeting the needs of individual members of the group and in developing, through close person-to-person relationships, the mutual understanding and good will the Japanese prized so highly.

5. At the end of each two-week period a combined meeting of all five groups was called for the purpose of coordinating the work, discussing common problems encountered, planning methods of attack, and, in general, to improve inter-group communication.

6. During the fourth and fifth weeks each group presented semi-final reports of their accomplishments and sought recommendations for the form in which the final reports were to be presented to the Ministry of Education, the sponsoring universities and IFEL headquarters.

The combined group reports presented at Hiroshima and at Tokyo are quite voluminous. They represent an enormous amount of labor on the part of the Japanese participants. The reports indicate that much has been learned and that much remains to be done before the Japanese teachers of mathematics will be satisfied that the subject they teach is making the contributions it can and should make to the achievement of specialized and general aims of education in a democracy.

The work of IFEL is not finished. It has merely begun. The Japanese IFEL groups have formed a "Mathematics-Education Research Association," which will continue to search for better ways of teaching more wisely selected subject matter, at more appropriate periods of children's growth and development, in a more interesting and democratic fashion. I am proud to be an honorary member of this dynamic organization.[3] The prospects for continued improvement in the mathematical education of Japan rest to a large extent with those IFEL professors of mathematics who visualize the complementary roles which mathematics and professional education must assume to produce the kind of teachers they need for new Japan.

V. Some accomplishments; some unsolved problems

A. *Accomplishments (general)*

1. Ultra-nationalistic and militaristic influences, as found in textbooks, curricula, teachers, and other personnel, have been removed (as far as can be expected) from the schools of Japan.
2. Compulsory school attendance for nine years is in effect. This covers the six-year elementary school and the three-year lower secondary school. Greater educational opportunity is thus provided for all children of all socio-economic groups.
3. The structural organization of the Japanese public schools has been simplified. The 6-3-3-4 plan is the legal organization today.
4. Co-education is practiced at all levels. This affords girls and young women the basis for equal oppor-

[3] In August 1955, Professor Snader returned to Japan as a consultant to the largest textbook publishing company in Tokyo. While there, the Japanese professors of Mathematics-Education held their reunions of the Hiroshima and Tokyo IFEL groups. Professor Snader was their guest speaker at these two national meetings.

tunities with boys and young men for higher education.

5. The control of education is decentralized. Formerly the *Mombusho* had supreme central control. The trend now is toward local and prefectural control.
6. The *Mombusho* now operates on a consultative and advisory basis. It seeks to develop local leadership rather than to dictate policies in an arbitrary manner.
7. The construction of new school buildings and the repair of bomb-damaged ones is carried on as fast as economic conditions permit.
8. The national support of public education is greatly improved. Teachers' salaries are now on a par with other comparable civil service positions. This may be the result of a strong teachers' union.
9. Textbooks and other instructional materials are being revised and improved. These materials are now written by individual authors, not by *Mombusho*. Recently a group of six men from the largest textbook publishing firm visited the U.S. to study means for improving their publications and to purchase modern printing equipment. They now have a new and thoroughly modern multi-million dollar plant near Tokyo equal to the best we have in the U.S.
10. Conferences, workshops, and group meetings for the improvement of education conducted along democratic lines are held with increasing frequency all over Japan.
11. New curricula and courses of study are being developed from kindergarten to the university and through the graduate schools.
12. Teaching methods are being studied and improved to provide for pupil participation, originality, and leadership development.
13. The Japanese written language is being simplified. Fewer *kanji* characters are being learned and instruction in *Romaji* is now begun in the elementary grades.
14. University standards are being up-graded. Autonomous university accrediting agencies are in operation.
15. Many of the separate schools and colleges have been consolidated under a single administrative control. This move was proposed by the U.S. Education Mission as an economy measure and to provide greater facilities for the development of professional leaders.
16. Special programs in such fields as industrial arts, health and physical education, radio, youth service, library science, and nursing education are being received with enthusiasm.

B. *Accomplishments* (in school mathematics)

1. An attempt is being made to develop mathematics-education as a science.
2. Psychologies of learning, teaching, motivation, etc. are being studied and applied in classroom work.
3. The curricula of the elementary school and lower secondary school are gradually being adjusted to the individual pupils rather than the pupils to the course of study. The individual Japanese pupil is becoming much more important than he ever was in pre-war Japan.
4. Aptitude and mental ability testing is now being used in guidance and for class groupings.
5. Arithmetic taught in the "experience curriculum" is accepted as sound procedure, but teachers are demanding additional class time for practice in arithmetic skills and fundamental operations after meanings and concepts are devel-

oped in realistic problem situations.

6. Since lower secondary school attendance is compulsory and the range of intelligence and mathematical aptitude among pupils is increasing, mathematics beyond the ninth grade is elective. (In the ninth grade pupils with low aptitude for mathematics usually do not elect the basic Analysis I course but a general mathematics course of typical American style.)
7. The selection and grade placement of mathematics content for the upper secondary school is being studied carefully. The universities expect the graduates from high school to have a working knowledge of calculus before they enter. They usually start their college work with advanced calculus.
8. Normal schools were reorganized into universities of education (one in each prefecture). Departments of education in which mathematics-education is taught have been established in all national universities. New positions, professors of mathematical education, have been created.
9. Summer sessions and in-service educational programs are doing much to up-grade the teaching of mathematics.
10. The training program for teachers of mathematics now includes a course in mathematics-education. This was not the case in pre-war Japan, except in the two higher normal schools of Tokyo and Hiroshima.

C. *Some unsolved professional problems*

1. Teacher education in the new national and public universities of Japan does not have the *facilities* (buildings, labs, shops, libraries, etc.), the *trained faculties*, nor the *professional programs* (except in mathematics-education) for top-flight work.
2. The new universities are not attracting students in sufficient numbers to prepare elementary and secondary school teachers to man the classrooms in the rapidly expanding free school system of Japan.
3. The departments of education in the large and famous national universities have not yet fully accepted their responsibility in the field of teacher education.

   Provisions need to be made for:
   a) financing the advanced training or re-training of staff members.
   b) financing graduate programs to prepare staff members for other colleges and universities, superintendents of schools, directors, principals, research specialists in education, etc.
4. In-service education requires:
   a) better staffs
   b) more comprehensive professional programs
   c) more easily accessible university service centers
5. The problem of preparing persons to conduct research and experiments in the schools of Japan—to find the solutions to their own professional problems—is a major one. A few weeks ago I received a report of mathematics-educational research being done in Japan. It is a very encouraging report.
6. The problem is acute in classes such as mathematics. It is not unusual to find 50–60 pupils in one class.
7. School textbooks and other instructional materials are costly. Pupils cannot afford to buy them. It was hoped that by the year 1952 the basic arithmetic and the Japa-

nese literature books would be furnished free. The Prefectural Board of Education and the Ministry of Education agreed to pay the bill.

8. While the teachers' salaries are supposedly on a par with other civil service employees, the salaries paid to teachers are much too low. Average salary for elementary school teachers for a twelve months' term is approximately $360; secondary school teachers about $400 for twelve months; college professors about $900.

This account is, of necessity, brief. In no sense is it complete. Further developments are taking place rapidly in Japan and new problems will most certainly arise in the future.

Japan's hope for the future rests with her schools. *She needs our help.* She appreciates the services we try to render. How can we help her democratize her schools and develop mutual and lasting friendships? The Japanese are anxious to be our friends. ***In the interests of peace and good will, shall we as teachers of mathematics accept the challenge?***

High Schools' science work termed "sad"

Must meet Russia's level: Strauss

Russia's apparent production of scientists and engineers at a faster rate than that prevailing in the United States is causing increasing concern among top Washington officials.

As one suggested remedy, Lewis L. Strauss, chairman of the Atomic Energy Commission, proposes that "every engineer and scientist in the country" volunteer to teach physics, chemistry, and mathematics a few hours a month in the high schools

He also suggests that colleges raise their minimum entrance requirements in physics and chemistry, thus forcing secondary schools to correct what he terms the "sad" level of scientific study in the high schools.

In line with one of Strauss' proposals, Brigadier General David Sarnoff, board chairman of Radio Corporation of America, urged last week that a "national educational reserve" be created.

"I have in mind," he said, "the release—with full pay for at least a year—of a reasonable number of men and women for teaching assignments in the local schools."

Another proposal now being discussed by some officials in Washington is establishment of a national science academy, akin to the military academies, where bright sliderule boys would receive a free federal education in return for giving the government their technical skills for a specified number of years.

President Barnaby C. Kenney of Brown University took the opposite tack in a speech here last week in which he deplored America's lack of "confidence" in her abilities.

"We cry in loud and piteous voices," he said, "that the Russians produced more scientists last year than we did, though we do not inquire whether these scientists are able or not."

Washington officials, however, do not tend to minimize the skills of Russian scientists.

Strauss says that the display of Soviet scientific knowledge at last summer's Geneva atomic conference "was sufficient to shatter any complacency we may have enjoyed in regard to our own imagination and ability."

He concludes that Russia now turns out "well-trained" and "highly competent" scientific personnel.

That, in brief, is the belief of responsible officials in the Eisenhower administration—and they're worried about it.—*Fletcher Knebel, Washington Bureau. Taken from the* Des Moines Sunday Register, *January 29, 1956.*

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

Area device for a trapezoid

by Emil J. Berger

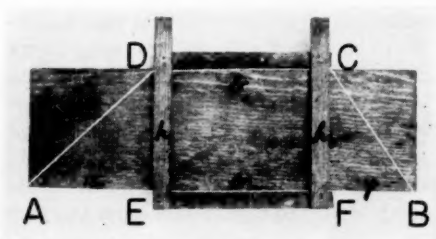


Figure 1

Illustrated in Figure 1 is a clever little student-made device which is useful in developing the usual formula for the area of a trapezoid ($S=\frac{1}{2}h[b+b']$). The device is in two parts: (1) a rectangular piece of fir $\frac{3}{4}'' \times 6'' \times 18''$; and (2) a double set of vertical runners which are attached to two narrow strips of wood that slide freely along the top and bottom. Lines AD and BC are pieces of white elastic. A look at the illustration will reveal that the shape of the trapezoid $ABCD$ may be varied as desired by sliding the double set of runners to the left or right.

The manner of using the device for the purpose suggested is indicated by the following development. The area S of the trapezoid is obviously equal to the sum of the areas of triangle ADE, rectangle $CDEF$, and triangle FBC. Thus,

$$S=(\tfrac{1}{2}hx+hb+\tfrac{1}{2}hy)$$

$$S=\tfrac{1}{2}h(x+2b+y)$$

$$S=\tfrac{1}{2}h\left[(x+b+y)+b\right].$$

Let $b'=(x+b+y)$.

Then, $S=\frac{1}{2}h(b+b')$.

How to draw a multiplication and division nomograph

by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Draw three equally spaced lines A, B, and C as indicated in Figure 2. Divide lines A and C into equal logarithmic scales. Divide line B into two logarithmic scales with each scale half as long as the scales on A and C. With a straightedge

connect any number on A with a number on C. The product of these two numbers will be found at the point on B located by the intersection of the straightedge ($A \cdot C = B$). This nomograph may also be used to find quotients ($B/A = C$).

When equal numbers are connected on scales A and C, the product on scale B is obviously the square of the number on A (or C). If the straightedge is placed perpendicular to line B, then for any number on B the reading on A (or C) is the square root of the number on B.

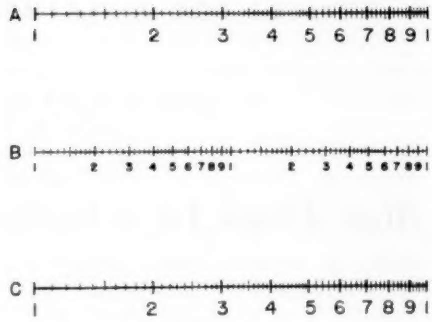


Figure 2

Letters to the editor

Continued from page 346

Yearbook.) Any device that helps in attaining this important objective is welcome. Even the narrow teaching practice properly attacked by the author as uninspiring can be made to yield greatly improved results if the "normal proposition" is varied by relaxing the given, by strengthening the given, and by relaxing and strengthening the conclusion. Ultimately the series of propositions obtained in this way would correspond to those obtained by invention and discovery. The approach is a matter of taste, certainly not a matter of thought control.

Very truly yours,
Charles T. Salkind

Dear Sir:

Ever since high school days I have been allergic to sentences such as "It can be proven that . . . ." "The proof is beyond the scope of this course." I came across such a sentence in Mr. Olmsted's article, "Parabola device" (THE MATHEMATICS TEACHER, October 1955, pp. 407–8). After showing that the equation of a certain family of straight lines is

$$\frac{x}{\alpha} + \frac{y}{1-\alpha} - 1 = 0$$

the author goes on: "In the calculus it is shown that the envelope of a family $f(x, y, \alpha) = 0$ is obtained by eliminating the 'parameter' α from the system of two equations $f(x, y, \alpha) = 0$ and $f_\alpha(x, y, \alpha) = 0$. . . . "

May I suggest a means not involving the calculus of finding the equations of envelopes of at least some families? The equation $f(x, y, \alpha) = 0$ relates the coordinates of any point through which a member or members of the family pass with the parameter α (in the present case the x-intercept), of the line(s) of the family through that point. If we solve that equation for α we shall in general find several solutions, one for each tangent to the envelope through the point. If the point chosen lies *on* the envelope, two such tangents will coincide. This will be indicated by a double solution.

Mr. Olmsted's equation may serve to illustrate the method: Solving the equation we have

$$\alpha^2 + (y - x - 1)\alpha + x = 0$$

hence

$$\alpha = \frac{-(y-x-1) \pm \sqrt{(y-x-1)^2 - 4x}}{2}$$

This will yield a double solution if and only if the discriminant is zero. Therefore, by setting the discriminant equal to zero we should obtain the equation of the envelope of the family. And we do indeed arrive at the same equation that Mr. Olmsted's excursion into the calculus has yielded.

I have tried both the method here described and the conventional method on a few families of curves whose equations were of the second degree in —. In every case tried, the non-calculus method was shorter as well as easier.

Sincerely yours,
CARL BERGMANN,
New York 31, New York

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Archytas' duplication of the cube

by R. F. Graesser, University of Arizona, Tucson, Arizona

Archytas of Taras was a Pythagorean and a friend of Plato. He flourished in the first half of the fourth century B.C. and was a paragon of accomplishments. Famous as a mathematician, as a statesman, and as a philosopher, he was also undefeated as a general. He devised one of the first and most remarkable solutions of the problem of duplicating the cube, i.e., constructing the edge of a cube whose volume is twice that of a given cube. His solution deserves to be more widely known. It was based on solid geometry and used, of course, pure geometry. What follows is a simplification, using spherical coordinates, of Archytas' solution. The reader unfamiliar with spherical coordinates is referred to Figure 1, in which P is any point with the rectangular coordinates (x, y, z) and the spherical coordinates (r, θ, ϕ). The distance, OP, or the radius vector of P, is r. Also θ is the colatitude of P and is the angle so labeled. Likewise ϕ is the longitude of P as labeled in the figure. It will be convenient to write ρ for $r \sin \theta$, where ρ is then the radius vector of the orthogonal projection of P on the xy plane.

Archytas' solution of the duplication problem consists in finding ρ for the point of intersection of three surfaces, viz., a circular cylinder, a circular cone, and a torus or anchor ring. (See Fig. 2 and Fig. 3.) We need the equations of these three surfaces. To obtain that of the cylinder, let P' in Figure 2 be any point

Figure 1

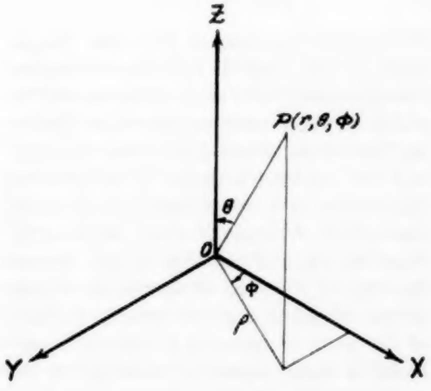
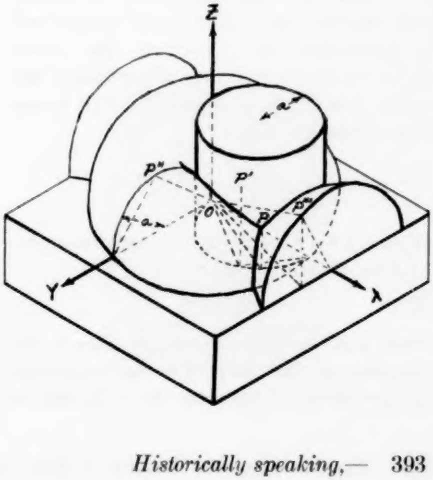


Figure 2



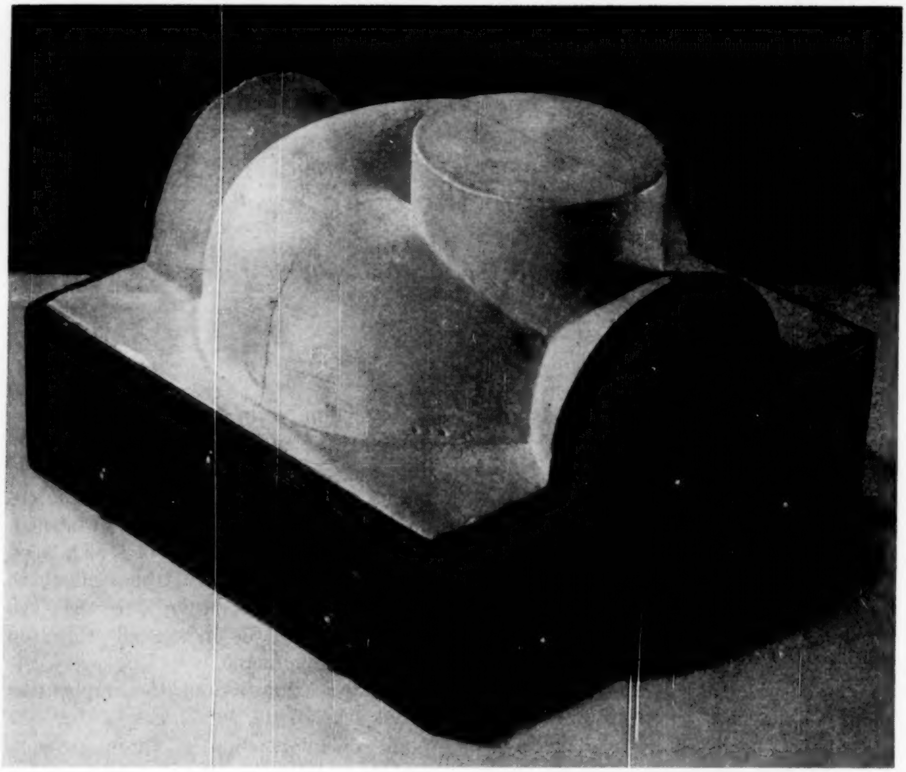


Figure 3

on it, and let a be its radius. Then, for all points on the linear element through P'

(1) $$\rho = 2a \cos \phi,$$

which is then the desired equation of the cylindrical surface. The torus, or anchor ring, has its inner radius equal to zero and is generated by revolving the circle $(y-a)^2+z^2=a^2$ in the yz plane about the z axis. Let P'' be any point on this torus. Then from the figure

$$2a \cos\left(\frac{\pi}{2}-\theta\right)=r,$$

or $2a \sin \theta = r$. Multiplying both members by r we obtain $2ar \sin \theta = r^2$, or

(2) $$2a\rho = r^2,$$

when ρ is written for $r \sin \theta$. This is the equation of the torus. Let the semiangle at the vertex of the cone be $\pi/3$, and let P''' be any point on the cone. The ρ and r of P''', when projected on the x axis, will have the same length which gives us $\rho \cos \phi = r \cos(\pi/3)$, or

(3) $$\rho \cos \phi = r/2.$$

This is the equation of the cone. Equations (1), (2), and (3) may now be solved simultaneously for the ρ of the point P in which all three surfaces intersect. Equating the values of $\cos \phi$ obtained from (1) and (3), we have $\rho/2a = r/2\rho$, or $r=\rho^2/a$. This value of r substituted in (2) gives $2a\rho=\rho^4/a^2$. Whence, $\rho^3=2a^3$, or $\rho=\sqrt[3]{2}a$. Thus the ratio of the edge of any cube to the edge of the cube of double its volume is that of radius a of the cylinder to the ρ of the point P common to the three surfaces—a most ingenious solution of the problem of duplicating the cube.

Editor's note

Professor Graesser's diagram, model, and modern symbolic proof of Archytas' procedure do an excellent job of making clear Archytas' hard-to-visualize solution of the duplication problem. Perhaps a few added remarks will increase its historical significance and its modern pedagogical utility.

Prior to Archytas, Hippocrates of Chios (*circa* 460 B.C.) had shown that the duplication of the cube problem could be solved if one could find two numbers "in continued mean proportion" between two given numbers. In modern notation this means: given numbers c and d, find x and y such that

$$\frac{c}{x}=\frac{x}{y}=\frac{y}{d} \tag{4}$$

That this is true follows from the fact that these proportions give the equations

$$x^2=cy \tag{5}$$

$$y^2=dx \tag{6}$$

$$xy=cd \tag{7}$$

Solving any two of these simultaneously gives $y=\sqrt[3]{cd^2}$. If now we take $c=2$ and $d=1$ we have $y=\sqrt[3]{2}$.

That Archytas' procedure was actually a means of solving this continued mean proportion problem is seen by noting that Professor Graesser's equations (2) and (3) may be written as $2a/r=r/\rho$ and $r/\rho=\cos\phi/\cos\alpha$ where α is the semiangle at the vertex of the cone. Now by multiplying numerator and denominator of the last fraction by $2a$, noting that $\rho=2a\cos\phi$ by equation (1), and equating equals we have

$$\frac{2a}{r}=\frac{r}{\rho}=\frac{\rho}{2a\cos\alpha}\,. \tag{8}$$

This we see is our version of Hippocrates' result, equation (4). If now we take $\cos\alpha=\frac{1}{2}$ or $\alpha=\pi/3$ as Professor Graesser did, formula (8) is specialized such that $\rho=\sqrt[3]{2}$ as he demonstrated.

The actual direct determination of the length ρ by Archytas' theoretically exact procedure can easily be seen to be very difficult if not impossible because of the three dimensional nature of the configuration. Menaechmus (*ca.* 350 B.C.) discovered that the continued mean proportion problem could be solved in two dimensions by finding the intersection of two parabolas or a parabola and an hyperbola as is shown by our equations (5), (6), and (7). This was the first appearance of the conic sections in Greek mathematics and perhaps in all mathematics. The later role of these curves in both mathematics and in the study of the physical world is another and well-known fascinating story.

The duplication problem and its solution by conics also played a role in Omar Khayyam's study of cubic equations. Several recent notes in THE MATHEMATICS TEACHER* furnish additional data on the duplication problem and its history and provide several good projects for either geometry, algebra, or trigonometry classes.—*P. S. J.*

*Juan A. Sornito, "Two Cube Root Curves," THE MATHEMATICS TEACHER, XLVII (December 1955), 560.

P. S. Jones, "Lill's Method," THE MATHEMATICS TEACHER, XLVI (January 1953), 35. This shows how a much more recent graphical device for solving polynomial equations in general really includes an ancient Greek mechanical device for solving the continued mean proportion problem.

Standard histories of mathematics, especially Greek mathematics, will also have more data, of course.

Mathematics on stamps

by H. D. Larsen, Albion College, Albion, Michigan

A list of stamps carrying portraits of mathematicians was published by the author in the November 1955 number of THE MATHEMATICS TEACHER. Important omissions in this list were kindly called to the author's attention by several readers. Also, during the year 1955 many new stamps were brought out in honor of mathematicians. All of these stamps are listed below and should be added to the list given previously.

A very striking stamp (one of a set of four) was issued in 1955 by Greece to com-

memorate the 2,500th anniversary of the founding of the Pythagorean School. This stamp, in black and white, pictures a 3-4-5 right triangle, a special case of what was perhaps the most important theorem due to the Pythagoreans.

Catalogue numbers and descriptions cited below are those in *Scott's Postage Stamp Catalogue, 1956.*

Avicenna (979–1037)
  Persia, 1954:
    B32. 1r+½r violet brown
    B33. 2½r+1½r blue
Buffon, Comte de (1707–1788)
  France, 1949:
    B241. 12fr+4fr purple
Carnot, Lazare (1753–1823)
  France, 1950:
    B251. 10fr+4fr lake
Copernicus, Nicholas (1473–1543)
  Poland, 1955:
    672. 40g violet on violet
  Russia, 1955:
    1752. 1r multicolored
Fragastero, Girolamo (1476–1533)
  Italy, 1955:
    685. 25l gray black and brown
LaPlace, Pierre Simon (1749–1827)
  France, 1955:
    B298. 30fr+9fr rose brown
Leibniz, Gottfried Wilhelm (1646–1716)
  East Germany, 1950:
  10N66. 24pf red
Monge, Gaspard (1746–1818)
  France, 1953:
    B279. 18fr+5fr dark blue
Pythagorean School (*fl.* 530–510 B.C.)
  Greece, 1955:
    582. 2d green
    583. 3½d intense black
    584. 5d plum
    585. 6d blue
Quevado, Leonardo Torres (1852–1939)
  Spain, 1955:
    C146. 50p bluish gray and black
Valéry, Paul Ambroise (1871–1945)
  France, 1954:
    B290. 30fr+10fr deep claret
Vega, Baron Jure (1756–1802)
  Jugoslavia, 1955:
    417. 50d deep claret

Mr. Conant looks at Europe

What are Europe's schools like after two world wars? Not much different than they were before 1914. The overwhelming majority of children rarely go beyond a common school education. A tiny minority of elite are prepared for secondary schools, which, in turn, prepare the intellectually able for the university. So says Dr. James Conant, U. S. ambassador to Germany.

Only 10 per cent of Europe's youth reach the high school. At the early age of 11, 12, or 13, European children are separated by an iron net of school grades and examinations into those who quit school and go to work and those who will enter the university, Dr. Conant said in interviews held while he was in New York City in January.

The high school in Europe, Dr. Conant said, "would delight the hearts of some of our dedicated subject-matter professors." The curriculum is heavy with ancient and foreign languages, stiff courses in mathematics, science, and history. Secondary-school students work hard and long. They memorize entire textbooks. For them there is in reality "full-time" education. "But for 90 per cent of those who are not destined for the secondary school," Dr. Conant explained, "there is no full-time education. Many millions of European children attend no more than five hours a week of schooling."—*Taken from* Edpress News Letter, *vol. 17, no. 9, January 24, 1956.*

"In mathematics, *equals* means the same as *alias.*"—Kershner and Wilcox, *The Anatomy of Mathematics.*

"These ideas we inherit . . . are never static. They are either fading into meaningless formulae, or are gaining power . . . by a more delicate apprehension."—Alfred N. Whitehead, *Science and the Modern World.*

Edited by Adrian Struyk, Clifton High School, Clifton, New Jersey

Odd and even—a game

by P. H. Nygaard, North Central High School, Spokane, Washington

There are two well-known mathematical recreations that start with a square framework consisting of three rows and three columns, each with three cells. In one of these the problem is to form a "Magic Square" by inserting into the spaces the digits 1 to 9 so that the sum in each row, column, and diagonal is exactly 15. The other is the game of "Tick-Tack-Toe," in which two players alternately place X's and O's in the spaces, the winner being the one who first gets either three X's or three O's in any row, column, or diagonal. A new game, "Odd and Even," which the author has tried out in his high-school classes, combines competition between two players, as in "Tick-Tack-Toe," and the use of numbers to add up to 15, as in the "Magic Square" puzzle.

The rules of the "Odd and Even" game are simple. Two pairs of crisscrossing parallel lines are drawn to form the familiar tick-tack-toe framework. One player has at his disposal the odd numbers 1, 3, 5, 7, 9; the other has the even numbers 2, 4, 6, 8. Since there is one more odd number than even number, the odd player always starts by placing one of his numbers in any of the nine spaces. The even player then puts one of his numbers in any space still open. The players continue to take turns placing their numbers in any open space. No number can be used more than once. The winner is the player who first places one of his numbers in any row, column, or diagonal so that the sum of his number and the two numbers already in that line is exactly 15. No win is scored unless there are three numbers in the line. If no one gets a line of 15, the game is a draw. In the next game the player who had the odd numbers should take the even numbers, so that the same player does not always start.

The procedure will be demonstrated by three specimen games, as diagrammed below:

6	7	2
5	3	4
9	1	8

4		9
8	5	2
		7

6	9	1
3	7	8
5	2	4

The order of play in the first game was 9, 6, 1, 8, 3, 4, 5, 2, 7. The odd player won the game with the line containing 6, 7, 2. The sequence in the second game was 5, 4, 7, 2, 9, 8. The even player won with the line 8, 5, 2. In the third game, which resulted in a draw, the order was 7, 2, 9, 8, 3, 4, 1, 6, 5. These illustrations are not intended to show best modes of play for either side.

Most games observed between high-school students seem to be draws. If the players are careless, either side seems to

have about equal chances. It is the author's opinion, however, that the odd player has the advantage if he plans his play several moves ahead. Perhaps exhaustive analysis will show that the odd player can always force a win. The author would like information regarding results or systems of play from anyone who gives "Odd and Even" some study and trial.

Errata notice

The table of Stirling II numbers published in the February 1956 issue of THE MATHEMATICS TEACHER (XLIX, 128–133) contains the following error: in column 24, line 3, for 23 read 32. That is,

$$_3S_{24}=4\ 70632\ 00806.$$

An obvious error also occurs on page 128 in equation (1) of the introduction. For the left-hand member read x^n instead of $x^{(n)}$.—*Editor.*

Have you read?

DUNNING, JOHN R. "The Best Jobs in Tomorrow's World," *Town Journal*, December 1955, pp. 20–21 and 67.

Math, chemistry, and physics may be *tough*, but those who can master them will get the best jobs in tomorrow's world. This statement is one all mathematics teachers should consider and bring before their students. The article is packed with information of interest to the potential scientist. He will want to know that only 20,000 were graduated last year when 40,000 were needed, that each had an opportunity for three to four jobs at over $400 per month. He will be interested to know that in several years his income will go to $15,000 or maybe $30,000, that over one-third of the presidents of 150 leading industries are engineers, and that he may specialize in many areas of engineering, such as chemical, civil, electrical, industrial, mechanical, metallurgical, and probably many others.

You will be interested in the criteria one may use to select the potential scientist. Even your students will want to know these criteria. I think this is a reading must for both teacher and student.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Lines from a red pencil

By Katharine O'Brien, Deering High School, Portland, Maine

At the end of the day no end to the toil—
Still tests to correct by the midnight oil.
I'm a perfect fright
By Friday night.

There are lots of jobs that are much more glamorous.
The salary question has long been clamorous.
Then tell me quick—
Why do I stick?

It's Willie—the way he devours the stuff.
Comes back for more—can't get enough.
Oh, what a boy!
My pride and joy.

If we tackle a problem that calls for a tussle,
The look on his face as he uses his muscle!
Deep down inside
I purr with pride.

Some day he'll rank as a research scholar.
(And I'll be down to my bottom dollar.)
I know it's silly,
But I teach for Willie.

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Rationalizing multiplication of decimal fractions

by John L. Marks, San Jose State College, San Jose, California

The importance of decimal fractions in modern life is increasing. They are needed for reasoning and problem solving in machine shops, technical and scientific laboratories, business offices, and testing rooms; they are used by adolescents and adults in do-it-yourself projects, in hot-rod building and repairing, in figuring and interpreting results of certain athletic events, and in many other activities.

Junior-high-school teachers have the responsibility not only of furnishing refresher and remedial activities for multiplication of decimal fractions, but also of providing new approaches to understanding of the algorism. Ample evidence is available to support the conclusion that drill in a given process without the development of the mathematical rationale of that process is inadequate. Further, repetition of the approach used when the topic was originally introduced is ordinarily ineffective both as a remedial experience or for refining mathematical meanings. Specifically, it should be recognized that few if any pupils, when entering the junior high school, have mature concepts of decimal fractions or a satisfactory understanding of algorisms.

As one outcome of the study of multiplication of decimal fractions, every pupil should be able to use *with understanding* the rule governing this operation: *Multiply as with integers and point off as many decimal places in the product as the combined number in the multiplier and multiplicand.* The ability to use this rule effectively in computation is temporarily achieved, but the skill is soon forgotten by many pupils. The desirability of and techniques for developing understanding of the law in the course of remedial work are often overlooked. The purpose here is to show briefly how the mathematical rationale for decimal multiplications may be developed through various estimation techniques and close attention to place value.

1. *Methods of estimating answers.* Estimation of answers prior to computation not only affords a check by revealing unreasonable results, but also provides confidence in the rule for placing the decimal point. Many decimal multiplications can be estimated by elementary methods, while others incorporate more mature concepts of place value or division by powers of ten. Whatever technique is employed, estimations should be performed mentally to be useful; hence, simplified procedures within the ability of most pupils are necessary.

a) Mental approximation by rounding off. Pupils should be guided in recognizing that, for purposes of estimating, parts of both multiplier and multiplicand which contribute least to the product can be discarded in arriving at factors that are easy to handle. Even relatively large adjustments to simplify both factors provide useful approximations. Thus, in the multiplication $.05 \times 42.7$ one estimation is $.1 \times 40 = 1/10$ of 40, or 4. Here, even doubling the multiplier yielded a suitable estimation, since possible exact answers might be: .2135, 2.135, 21.35, 213.5, etc., and the one of these closest to 4 is 2.135. A more refined method has the estimation preceded by "less than" or "more than." Thus, in the above example the correct thinking would be "the answer is less than $.1 \times 40$," since for purposes of estimation the multiplier was doubled while the multiplicand was decreased by less than 1/10.

It is readily observed that approximation may be less precise than is commonly realized and still be useful. For most decimal multiplications within the environment of the average person any of a wide choice of factors is adequate for locating the decimal point by estimation. Given 8.23×78.7, the factors may be rounded off as 10×100. These are good choices, although not necessarily those which provide the most accurate approximation, since they are the simplest numbers with which to compute and the estimated product (1000) makes it possible to eliminate incorrect answers from among the following possibilities: 6.47701, 64.7701, 647.701, or 6477.01.

b) Using division by ten and powers of ten. The procedure outlined under (*a*) also applies with factors less than one. Approximate solutions may be obtained by applying generalizations covering division by ten and powers of ten. In $.002 \times 58.7$ one thinks $.001 \times 50$ (interpreted as 1/1000 of 50) and moves the decimal point in 50 three places to the left to arrive at an estimate of .05. Or in $.041 \times 7.63$ one reasons: .04 or $7 = 4 \times 1/100$ of 7; this is 1/100 of 4×7, or 1/100 of 28; hence, the estimate is .28.

c) Establishing bounds between which products are located. Frequently it is possible to make two approximations, one stated "less than" and the other, "more than." These are upper and lower bounds for the product. For example, given 3.6×4.82, the pupil who thinks $3 \times 4 = 12$ recognizes his estimate is too small and knows the answer is more than 12. If he uses $4 \times 5 = 20$, he realizes the estimate is too large and reasons that the answer is less than 20. As a result he has established values 12 and 20 between which the exact answer falls. With experience he will come to observe that the factors seem closer to 4 and 5 than to 3 and 4, indicating the product is closer to 20 than 12.

The methods for estimating described above are within the abilities of most junior-high-school pupils. They are useful not only to develop confidence in computational procedures, but also to add variety in checking answers.

2. *Rationalizing multiplication of decimal fractions by converting to common fractions.* Any multiplication problem expressed using decimal fractions can be solved by converting the factors to common fractions. Examples illustrating the method are:

$$3 \times .2 = 3 \times 2/10 = \frac{3 \times 2}{10} = 6/10 = .6;$$

$$.3 \times 5.2 = 3/10 \times 5\ 2/10 = 3/10 \times 52/10$$

$$= 156/100 = 1.56.$$

Solutions in this form are not used for purposes of providing drill, although they do show a method of obtaining an answer. More important, however, they furnish an opportunity to discover the rule for placing the decimal point. Pupils, after completing the following table, have the necessary data needed to formulate the law.

Problem	Decimal places in multiplier	Decimal places in multiplicand	Answer (secured by using common fractions)	Number of decimal places in the product
3× .2	0	1	6/10= .6	1
3.1×4.71	1	2	$\frac{14601}{1000}$=14.601	3
.3×5.2	1	1	$\frac{156}{100}$= 1.56	2

The completion of these and other examples enables pupils to suggest the tentative hypothesis: The quantity in column two plus the quantity in column three equals the quantity in column five. Checking this theory with additional problems makes credible the mathematical law. It should be understood that no rule has been proved by this inductive or experimental method. However, investigating separate cases, developing hypotheses, and checking them is a means of verifying known laws and making them more plausible.

3. *Rationalizing multiplication of decimal fractions by means of place value concepts.* While studying the multiplication of decimal fractions, pupils review and refine their understanding of some important generalizations: ones×tenths =tenths; tenths×tenths=hundredths; tenths×hundredths=thousandths, etc. These and other related generalizations may be clarified by experimentation and recording of the data as follows:

Problem	Statement	Answer (using common fractions)	Decimal part of the answer expressed as	Statement of product
.21×3	hundredths times ones	63/100	hundredths	hundredths times ones= hundredths
.61× .3	hundredths times tenths	183/1000	thousandths	hundredths times tenths= thousandths

Rationalizing the rule for locating the decimal point in products is now possible using these place value concepts. In the example 4×2.163 the smallest units in the product result from the multiplication 4×3 thousandths; since this partial product is thousandths, the answer contains three decimal places. Likewise, in 21.67×39.4 the smallest numbers in terms of place value are 7 hundredths in the multiplier and 4 tenths in the multiplicand. Their product, thousandths, is the "small-

est" in the answer, so the decimal point is located three places from the right in the product. Developing an understanding of the principles described above is aided if pupils are encouraged to locate decimal points in the partial products and record the factors as shown in the example below:

$$\begin{array}{rl} 3.94 & \\ \times 2.8 & \\ \hline 3.152 = & .8\times 3.94 \\ 7.88\ = & 2\times 3.94 \\ \hline 11.032 = & 2.8\times 3.94 \end{array}$$

The thinking accompanying this operation is: tenths $\times$ hundredths = thousandths etc.

Other methods for locating the decimal points in products make use of generalizations relating to multiplication and division by ten and ten to a power. Multiplying by 6.21 is equivalent to a multiplication by 621 followed by a division by 100. Similarly, multiplying by 31.9 yields the same result as multiplying by 319 and dividing by 10. These facts lead to the solution of 6.21×31.9 as follows: $6.21\times 31.9 = (621 \div 100) \times (319 \div 10) = (621\times 319) \div 1000$. The operation is verbalized thus: "Multiply as if no decimals are present, and divide the answer by 1000." It is observed that dividing by 1000 moves the decimal point three places to the left in the product thereby justifying the algorism for this example.

A variation of the method just described compares all decimal multiplications to that problem with the decimal points removed. Thus, to complete $8.2\times .7$ one multiplies 82×7 obtaining a product of 574. But both factors in the original example are 1/10 as large, making the correct product 1/100 as large; hence, $8.2\times .7 = 5.74$ (1/100 of 574).

The ideas discussed above are useful in a refresher program when it is important that pupils do not lose sight of the mathematical "why" of the process. They provide a logical basis for rules, and thereby promote pupil understanding of the algorism.

An evaluation of the effectiveness of these procedures would be obtained by securing answers to these questions: Are pupils increasingly sensitive to the reasonableness of answers? Do they understand the reason for whatever rule they may use in "pointing off" products? Is the amount of repetitive practice necessary to complete learning reduced? Are skills and understandings transferable to new situations having similar and related elements? When pupils have forgotten the procedures normally used, do they substitute methods that secure the correct result or help them to recall the usual approach?

A junior-high-school program that achieves the outcomes sought from arithmetic instruction at that level must lead to affirmative answers to questions such as these. Each pupil must develop the ability to think with and about numbers as well as to improve his skill in computation. To this end, "discovery" and "thinking" activities such as outlined here must be an integral part of the experience of every pupil.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Mathematics and Nazism in retrospect . . .

Momentous events cannot be adequately appraised until viewed in historical perspective. It is now upwards of a quarter of a century ago that the preposterous myth of ARYAN mathematics was foisted upon a bewildered world. At the same time, the education of Nazi youth was distorted and twisted beyond belief, the perversion seeping down even into the area of mathematical instruction. Under the Nazis, so-called "free" or liberal professions such as the law, medicine, science, and teaching were rigidly controlled and regimented. Since such professional people were preoccupied with non-economic matters, their activities were inimical to totalitarianism. This partially explains the perverted educational policies, the enunciation of a "Nazi theory of physics," and the alleged "purification of mathematics."

In the early days of the Hitler regime when, among other things, science and mathematics were being "purified," it was decreed that "foreign" words denoting mathematical ideas were to be ruthlessly eliminated. All words of Latin, Greek, or other non-Germanic origin, as for example *Geometrie, Arithmetik, Quadrat, Kongruenz, Zylinder*, were to be replaced by equivalent native Teutonic forms. Thus, *Deckungsgleichheit* was to be substituted for *Kongruenz; Walze*, for *Zylinder; Spiegelung*, for *Symmetrie;* and so on. The proposed "purge" was exhaustive and met with almost complete success, on paper at least. Almost complete—for, although scores of such terms could be rendered into equivalent Germanic forms, yet the word "mathematics" itself proved to be a stickler. In most modern European languages the word for mathematics is easily recognized: *mathématique* (French), *matematica* (Italian), *matematicos* (Spanish), *Mathematik* (German), and so on. But when it came to purifying the word "Mathematik," ironically enough, no native German word could be found to take its place!

One of the many colossal and fantastic intellectual lies perpetrated by the Nazis was the claim of the superiority of "Nordic mathematics." It was dogmatically asserted that mathematics was a mirror of racial characteristics and an irrefutable index of racial traits. To show the desirability of "Aryan mathematics," it was urged that the greatest geometers—Steiner, Möbius, von Staudt, Riemann, Lie, and Grassmann—were all Teutonic. It was claimed that non-Aryans were notably lacking in spatial perception, that they under-valued the significance of intuition and tended to deny sensory evidence, and that they manifested an unreasonable liking for paradoxes. The Semitic peoples, it was further suggested, were lacking in qualities of imagination—hence, the preference for Abel over Jacobi, of Riemann over Einstein, and so on. In fact, the preposterous claims went to even more absurd lengths. It was brashly asserted that there existed a German mathematics and a Jewish mathematics—two entirely separate worlds. Jaensch's questionable psychological theory of types was freely drawn upon; thus, the French and Latins were supposedly the abstract-thinking S-type, whereas the Germans were of the I-type, who were most receptive to reality. As prototypes of the alleged German–Jewish antithesis in mathematics, they pointed to "Gauss the Saxon, and Jacobi the Oriental." Atten-

tion was called to the clear and limpid style of Gauss and his leanings toward intuition; Jacobi was portrayed as willfully abstract, diabolically clever, and possessed of ruthless egotism and intellectual arrogance.

The Jews were held responsible for the distinction between pure and applied mathematics. It was contended that Jewish thought exploited material already at hand, while Aryan thought was genuinely creative. The non-Aryan way led to the dehumanization of mathematics, to divorcement from nature and intuition: "There can be no complete mathematical domain independent of intuition and life; hence, the dispute over fundamentals now raging in mathematics is in reality a race struggle. Deep-rooted political implications mold the style of thought." So mathematics had to be freed from "the curse of sterile intellectualism." And since German mathematics was rooted in blood and soil, the State must support it. Furthermore, the greatest achievements of German mathematicians of both the past and of modern times attest the fact that "mathematics is beyond doubt a manifestation of national consciousness." And so it went, *ad nauseam*.

What utter nonsense! Truly ludicrous, were it not so fraught with tragic implications and dire consequences! Fortunately for mankind, the myth has now been exploded, but at a fearful price. One of the many lessons it should teach us is the need for a genuine appreciation of history—not the history of dynasties and wars and national heroes, but the history of the development of customs and inventions and ideas—in a word, an understanding of the growth and achievement exhibited by the human race as time marches on.

Again in retrospect, it is curious and perhaps instructive to note the perverted slant given to the mathematical education of youth under National Socialism in Germany during the decade or so preceding the outbreak of World War II. At that time many writers in Germany were insisting that the "organic theory of society" was the cornerstone of National Socialism; they strongly urged that mathematical instruction be used to develop the concept of the economic man with particular emphasis upon utility and thrift. All economic and political questions were to be discussed mathematically. The mathematics classroom was expected to help set in motion those "spiritual and inintellectual forces" in each pupil which would make him "an integral member of an organically organized society." In short, whereas in former times general culture was viewed as an acceptable goal of mathematical education, under the Nazis the sole task of all education, including mathematics, was to develop politico-economically-minded citizens. This particular emphasis upon social-political-economic problems was known as *Staatsbürgerkunde*.

Possibly a contributing factor to the development of the organic theory of society was the widespread popularity of youth movements, outdoor pageants, mass calisthenics, extensive hikes, and general participation in field sports (*Geländessport*). These recreational activities spread with phenomenal rapidity and were accompanied by such intriguing slogans as "*Kraft durch Freude*" and the like. This intense interest in field activities was reflected in textbooks and periodical literature in the form of material dealing with principles of dynamics and ballistics. It would have been surprising if the mathematical education of a people strongly unified through mass-mindedness and steeped in a social philosophy of force had not sooner or later reflected its concern with the application of science and mathematics to warfare and military preparedness (*Wehrkunst*). Schoolbooks of this period contained a wealth of material along these lines. One need only point out a few of the more typical and frequently reiterated exercises: the determination of angles and distances by indirect measurement; methods of triangulation; the de-

termination of inaccessible heights and distances by indirect measurement and trigonometry; the determination of the north-south direction by means of a watch; general problems of surveying, map-making, and engineering topology; the study of trajectories and ballistics; the timing of bombs and projectiles to explode at a desired time; the so-called "echo" problems, i.e., the determination of distances by measuring the time of the echo to return; and the perspective method of interpreting photographs of terrain taken from an airplane.

Totalitarian *Wehrwirtschaft* meant the organization of the nation's entire economic and social life along military patterns. There were no civilians, everybody was a "soldier" of the State. The essential purpose was to transform all social relationships into something akin to military ranks, i.e., of superiors and subordinates. In this way submission to authority and economic inequalities could be rationalized more readily, since they seemingly served military ends rather than economic purposes.

Whether popular interest in military applications of mathematics was a concomitant of the policy of *Wehrwirtschaft*, or whether it was deliberately fostered by Nazi leaders who (even before September 1939) were spreading the doctrine of *Lebensraum* and the notion that the outside world was bent on annihilating Germany, is beside the point. The fact remains that mathematical instruction in the schools consistently emphasized the relation of mathematics to military science, whether in the name of *Staatsbürgerkunde*, *Geländessport*, or *Wehrwirtschaft*.

du Pont Fellowships at the University of Chicago

The University of Chicago announces six graduate fellowships of $1,920 for students who wish to prepare for teaching chemistry or mathematics or physics in secondary schools. E. I. du Pont de Nemours and Company has granted the University funds for these fellowships to encourage able college graduates of both sexes to enter high-school mathematics and science teaching.

The du Pont fellows will register for science and/or mathematics courses, for professional courses in the teaching of the subject, and for apprentice teaching courses. Their program of study is designed to enable them to meet the requirements for the secondary-school teaching certificate issued by most states and to advance them toward a Master of Science degree.

Applications for fellowships for 1956–1957 must be made before May 15, 1956. Further information about the du Pont fellowships may be obtained from the Center for Teacher Education, University of Chicago, Chicago 37, Illinois.

I propose the establishment of a "National Educational Reserve" comprising qualified teachers in mathematics, physics, chemistry, engineering, and related subjects, to be drawn from the technological ranks of industry. I have in mind the release—and with full pay for at least a year—of a reasonable number of men and women for teaching assignments in their local schools. This unique Reserve could also mobilize those who have reached the retirement age, but whose knowledge and experience would make them inspiring teachers. In addition, it could include qualified people willing to volunteer their services to teach in night schools without giving up their industry jobs.—*Taken from* Our National Security, *an address delivered before the National Security Industrial Association by David Sarnoff, Chairman of the Board, RCA.*

• **POINTS AND VIEWPOINTS**

A column of unofficial comment

The National Council and the classroom teacher

by Marie S. Wilcox, Retiring President, NCTM

Children naturally like mathematics. This I believe! Certainly the majority of them do.

If more students are to study more mathematics, the classroom teacher must see that the student does not lose this natural eagerness to learn mathematics. As a matter of fact, the classroom teacher must see that the desire of the student to pursue his study of mathematics increases as he progresses in school.

This is a big order. It can neither be done by merely relating the uses of mathematics in engineering, nor by making pretty geometric constructions, nor by having an attractive bulletin board, although these things may have a place in the whole picture.

Young people will continue to enjoy the study of mathematics and will continue to enroll in courses in mathematics if they understand what they are studying and if they succeed.

The mathematics classroom teacher must, therefore, be a master teacher. She must have an excellent background in her subject matter. She must know what is happening in the mathematics classroom of capable teachers at grade levels above and below the level she is teaching. She should know what is new in methods of teaching mathematics and, certainly at the junior- and senior-high-school level, should have some knowledge of recent research in pure and applied mathematics.

The majority of members of the National Council are classroom teachers. Certainly, then, a major purpose of the Council should be to assist these members in doing an excellent job.

I believe that activities of the Council reflect this purpose. Meetings, other than the annual meeting, have been scheduled at times of the year when classroom teachers can attend. Council publications contain much that will assist the classroom teacher.

Latest developments in connection with Council publications are plans to publish and distribute a guidance leaflet to teachers of mathematics. The Board of Directors of the Council and the Board of Governors of the Mathematical Association of America approved the plans last summer. Dr. Irvin H. Brune of Iowa State Teachers College, representing the Council, and Dr. Alfred L. Putnam of the University of Chicago, representing the Association, are co-authors of the leaflet. The Council is soliciting financial assistance so that the leaflet may be distributed cost-free not only to members of our organization but to as many teachers of mathematics at the high-school level as it is possible to contact.

Also approved at the 1955 summer meeting of the Board were plans for a 1956 summer writing conference for the *Yearbook*, of which Dr. Phillip S. Jones is editor. This *Yearbook* is definitely meant

to be of assistance to the teacher in service, with information as to how central themes may be developed at various grade levels.

To assure the continuation of Council activities which will be of service to the classroom teacher, classroom teachers themselves must take an active part in the affairs of the Council. Classroom teachers have been and should continue to be officers and members of the Board, should accept important committee appointments, and should let their suggestions be known to the Board through the member of the Board representing their particular region or through the regular representative of their local affiliated group.

What's new?

BOOKS

Secondary

Mathematics Review Exercises (3rd ed.). David P. Smith, Jr. and Leslie T. Fagan. Boston: Ginn & Co., 1956. Cloth, vi+346 pp., $3.00.

Today's Geometry (4th ed.). Lee R. Spiller. Franklin Frey and David Reichgott. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1956. Cloth, 321 pp., $3.28.

Using Mathematics 7. Kenneth B. Henderson and Robert E. Pingry. New York: McGraw-Hill, Inc., 1956. Cloth, xi+436 pp., $2.96.

College

An Introduction to Linear Algebra. L. Mirsky. London: Oxford University Press, 1955. Cloth, xi+433 pp., $5.60.

An Introduction to Mathematics (rev. ed.). Lee Emerson Boyer. New York: Henry Holt & Co., 1955. Cloth, xvi+528 pp., $5.25.

Arithmetic for Engineers (5th ed.). Charles B. Clapham. London: Chapman & Hall, Ltd., 1955. Cloth, xiii+540 pp., 21s.

Arithmetic Its Structure and Concepts. Francis J. Mueller. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1956. Cloth, xv+279 pp., $5.50.

Calculus Differential and Integral. G. M. Petersen and R. F. Graesser. Ames, Iowa. Littlefield, Adams & Company, 1956. Paper, x+321 pp., $1.75.

Differential Equations, William Ted Martin and Eric Reissner, Cambridge, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956. Cloth, xi+260 pp., $5.50.

Elements of Business Mathematics for Colleges Llewellyn R. Snyder. New York: McGraw-Hill, Inc., 1956. Cloth, x+249 pp., $3.75.

Fundamental Concepts of Geometry, Bruce E. Meserve, Cambridge, Massachusetts, Addison-Wesley Publishing Company, Inc., 1955. Cloth, ix+321 pp., $7.50.

Geometry of Four Dimensions (Dover ed.). Henry Parker Manning. New York: Dover Publications, Inc., 1956. Paper, ix+348 pp., $1.95 (cloth, $3.95).

Modern Trigonometry. William A. Rutledge and John A. Pond. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1956. Cloth, xi+243 pp., $3.95.

Plane Trigonometry (3rd ed.). Alfred L. Nelson and Karl W. Folley. New York: Harper & Bros., 1956. Cloth, xi+330 pp., $3.50.

The Analytical Theory of Heat (Dover ed.). Joseph Fourier. Translated by Alexander Freeman. New York: Dover Publications, Inc., 1955. Paper, xxiii+466 pp., $1.95.

Topological Dynamics. Walter Helbig Gottschalk and Gustaf Arnold Hedlund. Providence, Rhode Island: American Mathematical Society, 1955. Cloth, vii+151 pp., $5.10.

Miscellaneous

Math. Can Be Fun (teacher ed.). Louis Grant Brandes. Portland, Maine: J. Weston Walch, Publisher, 1956. Paper, iv.+200 pp., $2.50.

Through the Mathescope, C. Stanley Ogilvy, New York, Oxford University Press, 1956. Cloth, vii+162 pp., $4.00.

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina, and Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois

BOOKS

Unified Algebra and Trigonometry, Elbridge P. Vance (Cambridge: Addison-Wesley Inc., 1955). ix+354 pp., $4.50.

This is an excellent text for one semester or two quarters work in college algebra and trigonometry. Emphasis is rightly placed upon analytic trigonometry with only 17 pages devoted to the solution of triangles.

A careful discussion in Chapter 3 of coordinate systems and the function concept paves the way for an elegant introduction of the *circular* functions in Chapter 4. Enough analytic geometry is developed so that a rigorous proof of the formula for cos $(\alpha-\beta)$ is easily presented.

Prior to the definitions of the inverse circular functions, an excellent discussion of the general concept of an inverse function is presented. An interesting treatment of addition of sine functions is followed by some stimulating remarks on harmonic analysis.

Four place tables of logarithms and trigonometric functions are included in the text. These certainly suffice for numerical purposes.

Considerably more than half the book is allocated to algebraic topics. These are almost invariably presented skillfully. The chapter on mathematical induction is admirably done. There is a fine treatment of determinants. Even the author's remarks on factoring are noteworthy. He carefully relates the factorization of polynomials to the factorization of integers and observes that factoring is *usually* restricted at the elementary stage to factoring polynomials with rational coefficients into prime polynomial factors free from irrational coefficients.

There are a few errors in the text. In the discussion on pages 41 and 42 of a one-dimensional coordinate system it is asserted that the 1-1 correspondence between the points on a line and the real numbers is effected by "associating every line segment with a real number which represents its length." This is incorrect. It is easy to *prove* from the postulates of geometry that every line segment has a length which is a real number (depending upon the unit segment chosen). This proof might well be incorporated into the text at this point. The key postulate which presents us with the 1-1 correspondence is the assumption that for *every number* there **is a** ***segment*** having the number for its length.

The proof, on page 148, of the theorem that the determinant of a square matrix is unchanged if corresponding rows and columns are interchanged is invalid. An elementary proof of this theorem is quite lengthy, but the ideas involved are of such significance that the text should be expanded by two or three pages at this point in order to develop the concepts needed for a proof.

The statement on page 150 of Property 6 for determinants is phrased: "If each element of any column (row) of a matrix is multiplied by the same number *m* and added to the corresponding element of another column (row). . . . " The formulation: "If to the elements of any column (row) are added the products of the corresponding elements of another column (row) by the same arbitrary number . . . " would seem preferable.

There is a large number of well-chosen exercises. Answers are given to odd numbered problems. The format is most attractive. It would be difficult to find a more satisfactory text.—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana.*

The Bequest of the Greeks, Tobias Dantzig (New York: Charles Scribner's Sons, 1955). Cloth, 191 pp., $3.95.

The Bequest of the Greeks is the first volume of a trilogy which has the title *Mathematics in Retrospect*, "Studies in the Evolution of Mathematical Thought and Technique Written for those who would cultivate Mathematics either as a Vocation or as an Avocation." The other volumes will have the titles *Centuries of Surge* and *The Age of Discretion* and will deal with the 17th and 18th centuries and the 19th century respectively.

The title of this volume indicates that the author is not writing a history of Greek mathematics *per se*. He is not dealing with what he aptly calls "mathematical archaeology." He is concerned with those things which modern mathematics has inherited and used, not with the oddments of the estate.

To suit the needs of the readers mentioned in the subtitle of the series, *The Bequest of the Greeks* is arranged in two sections. The first of these, "The Stage and the Cast," is addressed to

the general reader who realizes the importance of mathematics in contemporary thought and who wishes to learn more about its history. The second part, "An Anthology of the Greek Bequest," is directed to the student of mathematics with the aim of integrating his knowledge through this historical approach. It is safe to conclude that each will read both parts with profit.

Part I, "The Stage and the Cast," begins with a reference page showing a series of maps and accompanying data. This enables the reader to find the approximate date and locale of each of the individuals mentioned in the text, and it focuses attention on the contrast between the many centers of interest in mathematics during the period from Thales to Euclid with the concentration of that interest in the period from Menelaus to Diophantus.

Part I is a resumé of the development of the mathematical ideas which constitute the bequest. Here the reader must be on guard lest a finicky attitude color his appreciation of the volume. This is no place to debate the historicity of Thales or to argue whether a particular letter was written by Eratosthenes or by pseudo-Eratosthenes. Such preciosity works against our getting acquainted with an interesting scholar and hinders our appreciation of his thought.

In the seven chapters of Part I the author arranges his material under the heads: "On Greeks and Grecians," "The Founders," "On the Genesis of Geometry," "Pyramids," "Pentacles [the five-pointed star]," "The Pseudomath [circle squares and their ilk]," "The Interdiction [the restriction to the compasses and unmarked straight edge]." The last section will be of special interest to high-school teachers.

The chapter on pyramids will serve as an illustration of this section of the book. It begins with a quotation as does each chapter: "It is not that history repeats itself, but that historians repeat each other." Dr. Dantzig recapitulates the story that Thales, while in Egypt, was challenged to find the height of one of the pyramids and succeeded in doing so by measuring the shadow of the pyramid at a time when a man's shadow is equal to his height. The author notes that the story has been repeated again and again but without critical comment. He himself is not concerned with the authenticity of the tale but with its plausibility. He sets himself two questions: was this a problem which the Egyptians of Thales' day could solve on the basis of their knowledge; and, secondly, could Thales have done it on the basis of his knowledge. Dr. Dantzig indicates the inherent defects in the method usually shown and develops another in which the height of any object can be determined from two positions of the end of the shadows of the object and of a vertical post of known height. This proof satisfies the criterion of plausibility, for it comes within Thales' range of knowledge. As it depends only on the idea of similitude, it is possible that it also could have been developed by the Egyptians.

"The Anthology of the Greek Bequest" treats eight topics in considerable detail and indicates their later development. Thus, instead of leaving it to the student to discover what later mathematicians have done with various items, as for instance the Pythagorean relationship $c^2=a^2+b^2$, Mr. Dantzig has assembled this material in a connected account. It will be valuable.

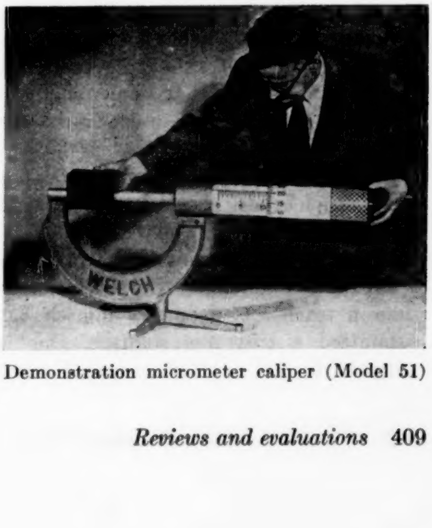
"The Anthology" is disappointing to the reader who keeps wondering when the author will come to some particularly favorite topic, as, for example, the conics of Apollonius. The Epilogue cares for this. Here the author says that he has postponed certain items so that they may be treated in connection with the time and place when they came into their fruition. This postponement makes the reader eager for the second volume, which the author says will follow "shortly."—*Vera Sanford, State Teachers College, Oneonta, New York.*

MODEL

Demonstration Micrometer Caliper (Model 51), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Blown-up model of micrometer caliper, $65.00.

This demonstration model is designed to show the principle of the micrometer screw in making precise measurements. The model is nearly a yard long when closed, is made principally of aluminum, weighs twelve pounds, has a yoke 19″ across, has a thimble 3½″ in diameter, and is 16½″ tall. The barrel and thimble are constructed of hard Bristol board tubing, to which printed scales are cemented and waterproofed. The smallest graduation of the linear scale on the barrel corresponds to the advance of the spindle caused by one revolution of the thimble. Two such graduations constitute the unit of measure indicated by the numerals on the barrel. The edge of the thimble is divided

Continued on page 411



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• NOTES FROM THE WASHINGTON OFFICE

Some unusual mail

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

As anyone acquainted with the growing program of the National Council of Teachers of Mathematics would expect, the Washington office receives a great volume and variety of mail. Although we have not attempted to keep an accurate count, the number of pieces of mail received and worked up by the office staff must be at least 30,000 per year.

This large volume of mail contains a surprisingly large number of letters that offer new "solutions" to old problems and "contributions" of various kinds. You may find it interesting to know something about this type of mail.

The most frequent solution submitted is that of trisecting the angle. Quite often high-school pupils submit such solutions. Some of them are rather ingenious. In most instances the error is quite transparent. In all of these cases we try to find time to analyze the work sufficiently to locate the basic flaw. Then we write the pupil, commending him on his interest and originality, pointing out the logical inconsistency in his proof, and suggesting references that he might like to read for further information about the trisection problem. We hope that some of these students are stimulated to continue their experiments in geometry.

All too frequently the pupils state that their work has been checked and approved by their teachers, and the teacher has encouraged the pupil to submit his proof for approval or publication. On several occasions a teacher of geometry himself has submitted a trisection solution. One is tempted to feel a bit discouraged when this happens. Every adequately prepared teacher of geometry should know that there are certain problems which cannot be solved with the traditional tools of unmarked straightedge and compasses.

Other letters that purport to contain new discoveries and ideas run to a great variety. Occasionally a solution to the problem of squaring the circle is submitted. One contribution consisted of what was supposed to be a detailed and complete outline of the field of mathematical knowledge. One interesting thing about the form of presentation was that the mathematical field was represented as completely closed and compartmentalized with no provision for the addition of new knowledge. The author of this project had hoped to expand his outline to include the entire range of human knowledge.

Frequently, "discoveries" of new methods, new teaching aids and gadgets, or new knowledge are presented with the confident belief that they will revolutionize the teaching of mathematics. Practically all of these contributions show an ignorance both of pure mathematics and of modern teaching methods and beliefs. Frequently they involve fantastic notions bordering on numerology. (Of course the mail does now and then bring from an informed teacher of mathematics an idea that really has merit. All such mail is forwarded promptly to the editors of our journals.)

One characteristic of nearly all of the mail described above is that the prime

concern of the authors seems to be to receive credit. Often, more than credit is wanted. The author feels that he is entitled to be paid liberally for his work, and he wants to sell it to the highest bidder. Occasionally we hear from a person who, although not wanting money for himself, is seeking a grant of funds that will enable him to finish the project that he has begun. The attitude of most of these persons is in sharp contrast with the scientific attitude and spirit of inquiry that have been responsible for the development of most of our knowledge.

Although this mail consists almost entirely of material that is either mathematically trivial or faulty, the fact that such mail exists probably represents a wholesome situation. It indicates that mathematics has a powerful and compelling attraction for many persons. It also indicates that many people who are attracted by mathematics never learn enough about it to really understand its nature. Incidentally, very little of the above mail comes from or through members of the National Council of Teachers of Mathematics. Except for the student mail, most of it comes from outsiders who are interested in our organization primarily as a channel through which they may publicize their ideas.

Reviews and evaluations

Continued from page 409

into 50 parts, so each thimble division is one-fiftieth of the pitch of the screw, or one-hundredth of the unit of measurement. This system corresponds to standard metric calipers having a half-millimeter pitch and reading to nearest 0.01 mm.

A setscrew in the thimble enables a zero adjustment to be made and also permits the removal of the entire thimble to expose the screw. The maximum opening between spindle and anvil is slightly more than 8 inches.

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"For education in a democratic society, which is also a changing society, the highest quality of performance of developmental tasks requires reflective thinking, or thinking through the situation before acting. The boy whose father and grandfather made buggy whips may without thinking much about it go into the buggy-whip business in an unchanging or a slowly changing society. But in a rapidly changing society the boy must think ahead, and determine, if he can, whether buggy whips will continue to be needed when he grows up."—*Taken from* Human Development and Education *by Robert J. Havighurst. New York: Longmans, Green and Company, 1953. Chapter 12, page 169.*

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Sixteenth Summer Meeting

University of California, Los Angeles Campus

August 19, 20, 21, 22, 1956

The program which is now being planned for this summer meeting of N.C.T.M. in Los Angeles will include general sessions addressed by nationally known speakers, a banquet, a luncheon, and many sectional meetings. These meetings should be of interest to teachers of elementary arithmetic, and junior and senior high school mathematics, as well as to teachers of junior and senior college mathematics. Special sections will also deal with aspects of teacher education in mathematics.

Complete programs will later be mailed to members of N.C.T.M. In the meantime, further information may be had from Dr. Clifford Bell, University of California, Los Angeles Campus, Los Angeles, Calif.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH NEA
July 2, 1956
Portland, Oregon
Lesta Hoel, Public Schools, Portland, Oregon

SUMMER MEETING
August 19–22, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

CHRISTMAS MEETING
December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

ANNUAL MEETING
March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

Other professional dates

Mathematics Institute
June 4–15, 1956
University of Oklahoma, Norman, Oklahoma
J. O. Hassler, Department of Mathematics, University of Oklahoma, Norman, Oklahoma

Ninth Annual Workshop for Teachers of Mathematics
June 18–30, 1956
Indiana University, Bloomington, Indiana
Philip Peak, School of Education, Indiana University, Bloomington, Indiana

Mathematics Institute (Sponsored by Iowa State Teachers College and the National Science Foundation)
June 18–July 28, 1956
Iowa State Teachers College, Cedar Falls, Iowa
H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

California Conference for Teachers of Mathematics
June 20–July 3, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

Seventh Annual Mathematics Institute of Louisiana State University
June 24–29, 1956
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

Workshop in Visual Aids in Mathematics for Junior-Senior High School
June 25–July 13, 1956
State Teachers College, Millersville, Pennsylvania
George R. Anderson, State Teachers College, Millersville, Pennsylvania

Institute on Secondary and Collegiate Mathematics
July 2–August 11, 1956
Williams College, Williamstown, Massachusetts
Donald E. Richmond, Department of Mathematics, Williams College, Williamstown, Massachusetts

Workshop in Mathematics Education for Senior High, Junior High, and Elementary School Teachers and Administrators
August 6–25, 1956
Northwestern University, Evanston, Illinois
E. H. C. Hildebrandt, 221 Lunt Building, Northwestern University, Evanston, Illinois

Eighth Annual Mathematics Institute of the Association of Teachers of Mathematics in New England
August 16–23, 1956
Williams College, Williamstown, Massachusetts
Barbara B. Betts, 138 Norfolk Avenue, Swampscott, Massachusetts

Fourth Annual Mathematics Institute of the Florida Council of Teachers of Mathematics
August 27–28, 1956
Florida State University, Tallahassee, Florida
Joe Hooten, Florida State University, Tallahassee, Florida

Call for nominations

The new Committee on Nominations and Elections is already making plans for the next election in 1957. Officers to be elected in this election will be as follows: Vice-President for Junior High School, Vice-President for College, and three Directors.

The committee is anxious to receive your suggestions for candidates. If you know of able individuals who should be considered for office, please submit their names now, with full information about their qualifications, to the new chairman of the Committee on Nominations and Elections, F. Lynwood Wren, George Peabody College for Teachers, Nashville Tennessee.

The members of the nominating committee are:

Alice Hach	John Mayor
Janet Height	Beth Patton
Clark Lay	Phillip Peak
Zeke Loflin	

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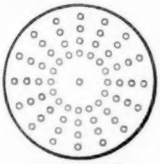

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